

The University of British Columbia

Final Examination - April 21st, 2017

Mathematics 215/255

Time: 2 hours

Last Name _____ First _____ Signature _____

Student Number _____

Section (circle one) Roxanas (215:201) / Coombs (215:202) / Rahmani (255:201)

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- **Show your work.** A correct answer without intermediate steps will receive no credit.
- This exam has 13 numbered pages, including this cover and a table of Laplace transforms.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		10
2		10
3		10
4		10
5		5
6		10
7		10
8		15
9		10
10		10
Total		100

Problem 1 (10 Points) Find all values of a for which the following problem is exact. Justify your answer. You do not have to solve the problem.

$$x^2 + y^2 + ay(x + 1) \frac{dy}{dx} = 0$$

$a =$

Problem 2 (10 Points)

a. Solve the initial value problem for $y(t)$:

$$\frac{dy}{dt} = y^2 \quad y(0) = 0.$$

$y(t) =$

b. Is your solution to this problem unique? Explain.

Problem 3 (10 Points)

a. Solve the initial value problem for $y(t)$. Show your work.

$$y'' + \pi^2 y = \sin(\pi t), \quad y(0) = 0; \quad y'(0) = 0.$$

$y(t) =$

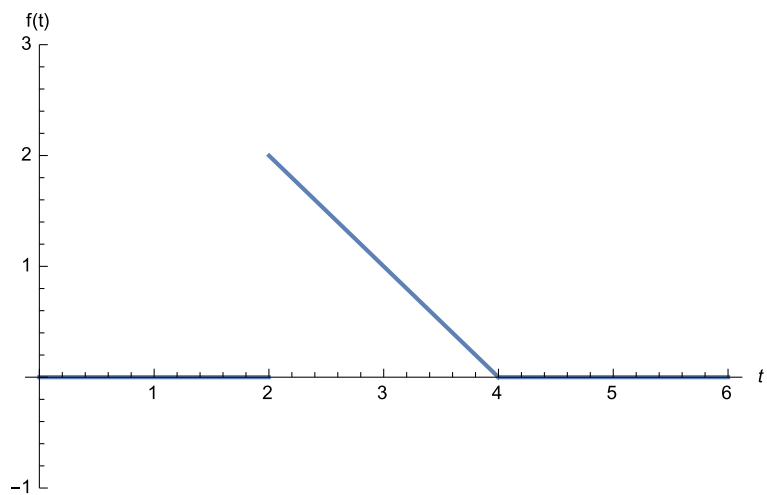
b. Draw a graph of the solution. Label axes so that the behaviour of the solution is clear.

Problem 4 (10 Points) Find the inverse Laplace transform of the indicated function. Show your work.

$$F(s) = \frac{(s + 3)e^{-4s}}{s^2 - 3s + 4}$$

$f(t) =$

Problem 5 (5 Points) Consider the function $f(t)$ in this plot:



Write a one-line expression for this function using unit step functions (Heaviside functions). Express the unit step function as $\mathcal{U}(t)$.

$f(t) =$

Problem 6 (10 Points) Solve the initial value problem using Laplace transform. Answers obtained using any other method will not receive any points. Show your work.

$$y'' + 16y = \begin{cases} t & 0 \leq t \leq 1 \\ 0 & 1 < t < \infty \end{cases} \quad y(0) = y'(0) = 0.$$

$$y(t) =$$

Problem 7 (10 Points) Consider the nonlinear system

$$x' = 3x(3 - y)$$

$$y' = 2y(2 - x).$$

- a. Find all the critical points of this system of differential equations.

Critical points:

- b. Sketch the phase portrait of the linearization of the system at each critical point. Plot the eigendirections carefully, and specify the type (node, saddle, spiral, center; stable, unstable).

Problem 8 (15 Points) The following Matlab script solves a system of first order ODEs with particular initial conditions.

```
m=1; c=0.2; k=0.1;
F0=1.0; omega=0.3;
ic=[0,1]; %initial conditions

xl=2; yl=2;
[X1, X2] = meshgrid(0:xl/200:xl,0:yl/200:yl);

F1=@(x1,x2) x2;
F2=@(x1,x2) -c/m*x2-k/m*x1;

tr=[0 100]; % time range of integration

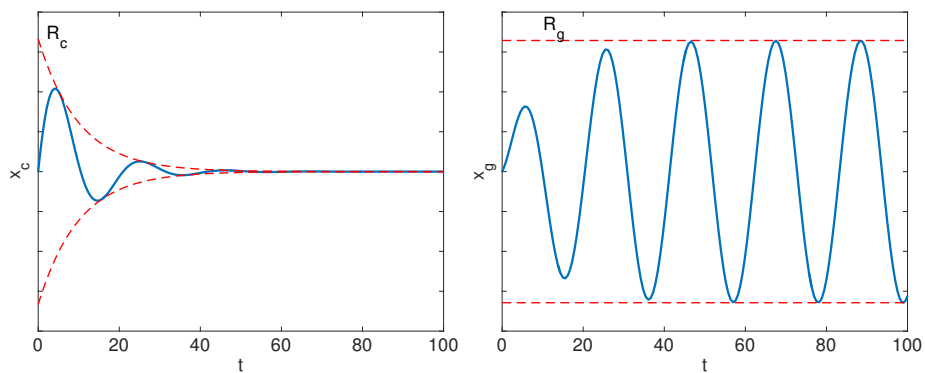
f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))+F0*cos(omega*t)];
[tout, xout]=ode45(f,tr,ic);
```

(a) Write down the system of first order ODEs and corresponding initial values.

(b) This system was derived by re-writing a single 2nd order initial value problem as a system. What is that problem? Don't forget to specify the initial conditions.

Problem 8, continued

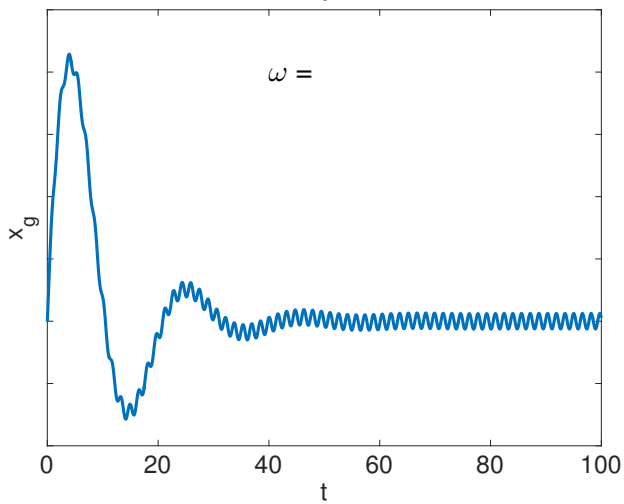
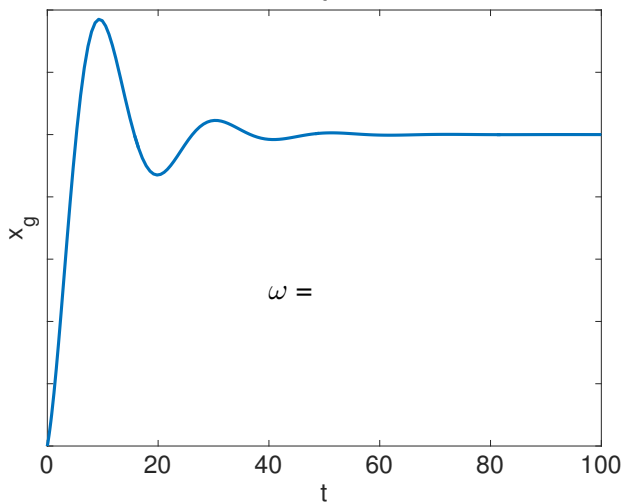
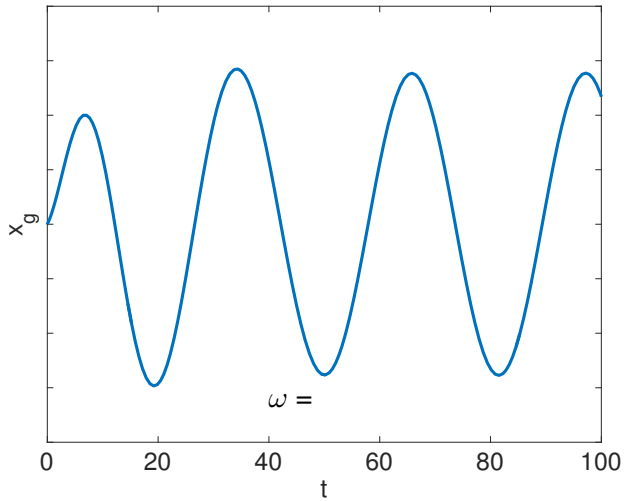
In the system above, x_1 can be interpreted as the displacement of a mass-spring system from its equilibrium position, and x_2 as the velocity associated with the motion of this system. Plots of the homogeneous (complementary) and general solutions of the oscillations of the mass-spring system described in the code above are given below.



(c) Find the amplitude of the steady-state oscillations, R_g .

Problem 8, continued

(d) The oscillations of the system given in the code due to forcing functions with different frequencies: (i) $\omega = 0$, (ii) $\omega = 0.2$, and (iii) $\omega = 4$ are plotted below. Match each plot with its corresponding forcing frequency and fill in " $\omega =$ " on each plot.



Problem 9 (10 Points) The motion of a particle with position $x(t)$ is governed by the following equation:

$$x'' + \kappa x' + x = 0.$$

The particle has initial position $x(0) = -1$ and κ is a positive constant.

- a. Suppose $\kappa = 2$. The particle is observed to pass through the origin ($x = 0$) exactly once. What does this tell you about the possible values of the initial velocity?

- b. Is it possible to select $\kappa > 0$ and an initial velocity so that the particle passes through the origin exactly two times? Explain your answer.

Problem 10 (10 Points) Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(-ay^2 + 2by + c), \quad y(0) = y_0,$$

for some real constants a, b, c .

a. Assume $b^2 + ac > 0$. Find $\lim_{t \rightarrow \infty} y(t)$ for all possible values of y_0 . You need to fully justify your answer.

b. Repeat the previous question for the same initial value problem, but now assume $a < 0$ and $b^2 + ac < 0$.

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\frac{1}{s}$ (1)	te^{at}	$\frac{1}{(s-a)^2}$ (13)
$e^{at}f(t)$	$F(s-a)$ (2)	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ (14)
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$ (3)	$e^{at} \sin kt$	$\frac{k}{(s-a)^2 + k^2}$ (15)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$ (4)	$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2 + k^2}$ (16)
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$ (5)	$e^{at} \sinh kt$	$\frac{k}{(s-a)^2 - k^2}$ (17)
$f'(t)$	$sF(s) - f(0)$ (6)	$e^{at} \cosh kt$	$\frac{s-a}{(s-a)^2 - k^2}$ (18)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$ (7)	$t \sin kt$	$\frac{2ks}{(s^2 + k^2)^2}$ (19)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$ (8)	$t \cos kt$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ (20)
t^n ($n = 0, 1, 2, \dots$)	$\frac{n!}{s^{n+1}}$ (9)	$t \sinh kt$	$\frac{2ks}{(s^2 - k^2)^2}$ (21)
$\sin kt$	$\frac{k}{s^2 + k^2}$ (10)	$t \cosh kt$	$\frac{s^2 - k^2}{(s^2 - k^2)^2}$ (22)
$\cos kt$	$\frac{s}{s^2 + k^2}$ (11)	$\delta(t - t_0)$	e^{-st_0} (23)
e^{at}	$\frac{1}{s-a}$ (12)		

Trig identities

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned}$$

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