The University of British Columbia

Final Examination - April 21st, 2017

Mathematics 215/255

Time: 2 hours

Last Name ____________________ **First** ______________ **Signature** ____________________

Student Number _____________________

Section (circle one) Roxanas (215:201) / Coombs (215:202) / Rahmani (255:201)

Special Instructions:

- Books, notes, cellphones and calculators are **not** allowed.
- Use the reverse side of each page if you need extra space.
- **Show your work**. A correct answer without intermediate steps will receive no credit.
- This exam has 13 numbered pages, including this cover and a table of Laplace transforms.

1		10
2		10
3		10
4		10
5		5
6		10
7		10
8		15
9		10
10		10
Total		100

Problem 1 (10 Points) Find all values of a for which the following problem is exact. Justify your answer. You do not have to solve the problem.

$$x^2 + y^2 + ay(x+1)\frac{dy}{dx} = 0$$

$a =$

Problem 2 (10 Points)

a. Solve the initial value problem for $y(t)$:

$$\frac{dy}{dt} = y^2 \qquad y(0) = 0.$$

$y(t) =$

b. Is your solution to this problem unique? Explain.

Problem 3 (10 Points)
a. Solve the initial value problem for $y(t)$. Show your work.

$$y'' + \pi^2 y = \sin(\pi t), \qquad y(0) = 0; \quad y'(0) = 0.$$

$y(t) =$

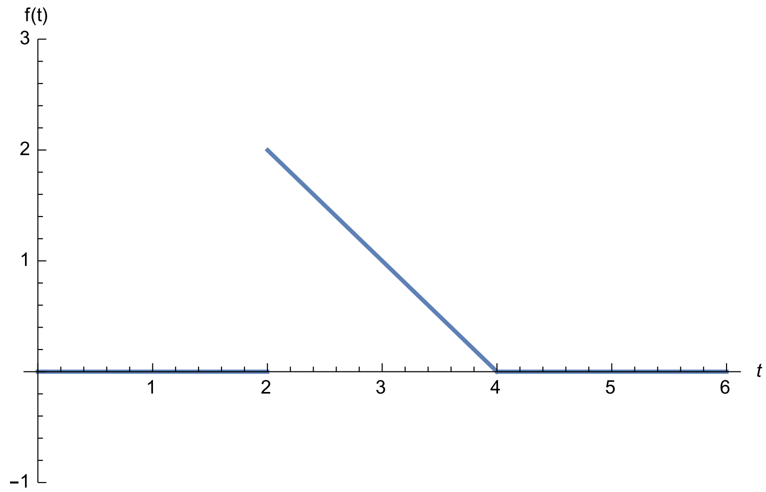
b. Draw a graph of the solution. Label axes so that the behaviour of the solution is clear.

Problem 4 (10 Points) Find the inverse Laplace transform of the indicated function. Show your work.

$$F(s) = \frac{(s+3)e^{-4s}}{s^2 - 3s + 4}$$

$f(t) =$

Problem 5 (5 Points) Consider the function $f(t)$ in this plot:



Write a one-line expression for this function using unit step functions (Heaviside functions). Express the unit step function as $\mathcal{U}(t)$.

$f(t) =$

Problem 6 (10 Points) Solve the initial value problem using Laplace transform. Answers obtained using any other method will not receive any points. Show your work.

$$y'' + 16y = \begin{cases} t & 0 \le t \le 1 \\ 0 & 1 < t < \infty \end{cases} \qquad y(0) = y'(0) = 0.$$

$y(t) =$

Problem 7 (10 Points) Consider the nonlinear system

$$x' = 3x(3 - y)$$
$$y' = 2y(2 - x).$$

a. Find all the critical points of this system of differential equations.

Critical points:

b. Sketch the phase portrait of the linearization of the system at each critical point. Plot the eigendirections carefully, and specify the type (node, saddle, spiral, center; stable, unstable).

Problem 8 (15 Points) The following Matlab script solves a system of first order ODEs with particular initial conditions.

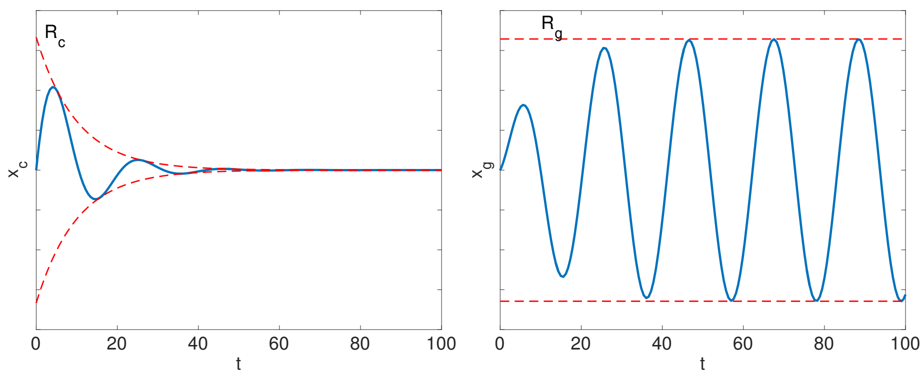
```
m=1; c=0.2; k=0.1;
F0=1.0; omega=0.3;
ic=[0,1]; %initial conditions

xl=2; yl=2;
[X1, X2] = meshgrid(0:xl/200:xl,0:yl/200:yl);

F1=@(x1,x2) x2;
F2=@(x1,x2) -c/m*x2-k/m*x1;

tr=[0 100]; % time range of integration

f=@(t,x) [F1(x(1),x(2)); F2(x(1),x(2))+F0*cos(omega*t)];
[tout, xout]=ode45(f,tr,ic);
```

(a) Write down the system of first order ODEs and corresponding initial values.

(b) This system was derived by re-writing a single 2nd order initial value problem as a system. What is that problem? Don't forget to specify the initial conditions.

Problem 8, continued

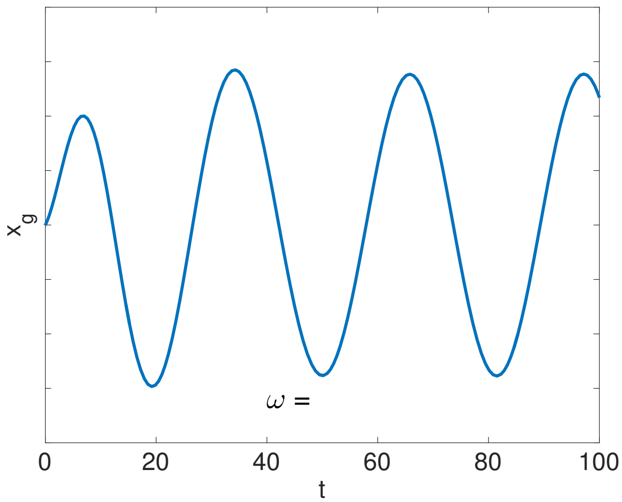
In the system above, x_1 can be interpreted as the displacement of a mass-spring system from its equilibrium position, and x_2 as the velocity associated with the motion of this system. Plots of the homogeneous (complementary) and general solutions of the oscillations of the mass-spring system described in the code above are given below.

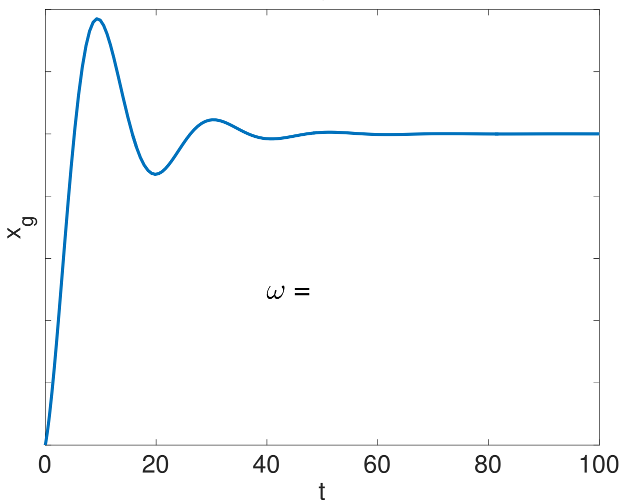


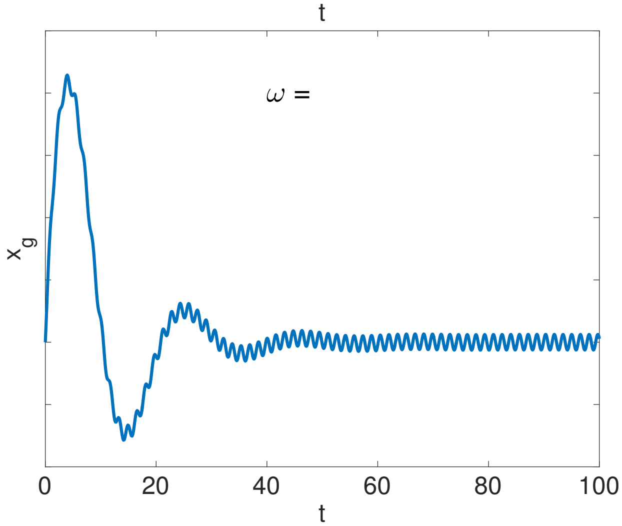
(c) Find the amplitude of the steady-state oscillations, R_g.

Problem 8, continued

(d) The oscillations of the system given in the code due to forcing functions with different frequencies: (i) $\omega = 0$, (ii) $\omega = 0.2$, and (iii) $\omega = 4$ are plotted below. Match each plot with its corresponding forcing frequency and fill in "$\omega =$ " on each plot.







Problem 9 (10 Points) The motion of a particle with position $x(t)$ is governed by the following equation:

$$x'' + \kappa x' + x = 0.$$

The particle has initial position $x(0) = -1$ and κ is a positive constant.

a. Suppose $\kappa = 2$. The particle is observed to pass through the origin ($x = 0$) exactly once. What does this tell you about the possible values of the initial velocity?

b. Is it possible to select $\kappa > 0$ and an initial velocity so that the particle passes through the origin exactly two times? Explain your answer.

Problem 10 (10 Points) Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(-ay^2 + 2by + c),\ y(0) = y_0,$$

for some real constants a, b, c.

a. Assume $b^2 + ac > 0$. Find $\lim_{t\to\infty} y(t)$ for all possible values of y_0. You need to fully justify your answer.

b. Repeat the previous question for the same initial value problem, but now assume $a < 0$ and $b^2 + ac < 0$.

END OF EXAM

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(5)
$f'(t)$	$sF(s) - f(0)$	(6)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	(7)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(8)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(9)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(10)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(11)
e^{at}	$\dfrac{1}{s-a}$	(12)

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
te^{at}	$\dfrac{1}{(s-a)^2}$	(13)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(14)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(15)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(16)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(17)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(18)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(19)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(20)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(21)
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$	(22)
$\delta(t-t_0)$	e^{-st_0}	(23)

Trig identities

$$\begin{aligned}\sin(A \pm B) &= \sin A\cos B \pm \sin B\cos A\\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B\end{aligned}$$