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The University of British Columbia

MATH 215/255, Sections 101–105

Final Exam – December 2014

Family Name \_\_\_\_\_ Given Name \_\_\_\_\_

Student Number \_\_\_\_\_ Signature \_\_\_\_\_

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No notes nor calculators.

**Rules Governing Formal Examinations:**

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  - (b) Speaking or communicating with other candidates;
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers, and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

| Problem | Points | Score |
|---------|--------|-------|
| 1       | 8      |       |
| 2       | 12     |       |
| 3       | 10     |       |
| 4       | 13     |       |
| 5       | 8      |       |
| 6       | 12     |       |
| 7       | 10     |       |
| 8       | 12     |       |
| 9       | 15     |       |
| Total:  | 100    |       |

1. Solve the initial value problem

$$(3xy + y^2) + (x^2 + xy)y' = 0, \quad y(1) = 2.$$

(3 points) (a) Verify that  $\mu(x) = x$  is an integrating factor, that is,  $x(3xy + y^2)dx + x(x^2 + xy)dy = 0$  is exact.

(5 points) (b) Solve the initial value problem.

2. A second order chemical reaction can be modeled by the equation

$$\frac{dy}{dx} = \alpha(y - p)(y - q),$$

where  $\alpha, p$  and  $q$  are constants

(4 points) (a) Assume that  $\alpha > 0$  and  $p > q > 0$ . Find equilibrium points and classify stabilities of these equilibrium points.

(4 points) (b) Assume that  $\alpha = 1, p = 0, q = 1$  and  $y(0) = -1$ , solve the initial value problem and determine the limiting value of  $y(x)$  as  $x \rightarrow \infty$ .

- (4 points) (c) Assume that  $\alpha = 1$ ,  $p = q = 0$  and  $y(0) = 1$ . Use Euler's method to approximate  $y(2)$  with the step size  $h = 1$ .

(5 points) 3. (a) Solve  $y'' + 4y' + 13y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 5$ .

(5 points) (b) Find a particular solution of

$$y'' + 3y' = e^{-3t}$$

using **either** the method of undetermined coefficients, **or** the method of variation of parameters.

4. Consider a vibrating system described by the initial value problem

$$u'' + cu' + 4u = \cos 2t, \quad u(0) = 0, \quad u'(0) = 2.$$

where  $c > 0$  is the damping coefficient.

- (5 points) (a) Find the steady periodic part of the solution (the part of the solution which remains as  $t \rightarrow \infty$ ) of this problem, and find its amplitude. Do not find the transient part.

- (2 points) (b) Let  $A(c)$  denote the maximum amplitude of the steady state solutions of the systems

$$u'' + cu' + 4u = \cos \omega t, \quad u(0) = 0, \quad u'(0) = 2$$

among all possible  $\omega > 0$ . What happens to  $A(c)$  as  $c \rightarrow 0_+$ ? Explain why.

*Hint.* You do not need to solve  $A(c)$  explicitly.

(6 points) (c) Find a particular solution of

$$y'' + y = \frac{1}{\cos t}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

*Hint:*  $\int \tan t \, dt = -\ln |\cos t|$ ,  $\int \cot t \, dt = \ln |\sin t|$

- (8 points) 5. Use the Laplace transform to solve the system  $x''(t) + 2x'(t) + 2x(t) = 2$  with initial conditions  $x(0) = 0$ ,  $x'(0) = 0$ .



(6 points) 6. (a) Solve the system  $x''(t) + 4x(t) = \delta(t - 2)$  with initial conditions  $x(0) = 0$  and  $x'(0) = 0$ .

(6 points) (b) Find the Laplace transform of

$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2. \end{cases}$$

(7 points) 7. (a) Find general solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} p & 4 \\ -1 & p \end{bmatrix},$$

assuming that  $p$  is real.

(3 points) (b) Describe the behaviour of the system (do not draw phase portrait) for all possible real values of  $p$ .

(12 points) 8. Solve

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} - \begin{bmatrix} 0 \\ 3t + 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

9. Consider the nonlinear system

$$\frac{dx}{dt} = x^2 - \frac{y^2}{2} + 1, \quad \frac{dy}{dt} = -4x - 2y. \quad (1)$$

(5 points) (a) Find the equilibria (critical points) for the system (1).

(6 points) (b) Find the Jacobian (partial derivative) matrix for the system (1), compute the linearized system at each equilibrium, and compute the eigenvalues for each of the coefficient matrices.

- (4 points) (c) Identify the equilibrium of (1) at which the linearized system (not system (1)) has a center. Classify the other equilibrium and indicate whether it is (asymptotically) stable or unstable in system (1).

### Table of Laplace transforms

| $f(t) = \mathcal{L}^{-1}\{F(s)\}$    | $F(s) = \mathcal{L}\{f(t)\}$                    |
|--------------------------------------|---|
| 1. 1                                 | $\frac{1}{s}, \quad s > 0$                      |
| 2. $e^{-at}$                         | $\frac{1}{s+a}, \quad s > -a$                   |
| 3. $t^n$ , $n$ positive integer      | $\frac{n!}{s^{n+1}}, \quad s > 0$               |
| 4. $\sin(at)$                        | $\frac{a}{s^2 + a^2}, \quad s > 0$              |
| 5. $\cos(at)$                        | $\frac{s}{s^2 + a^2}, \quad s > 0$              |
| 6. $\sinh(at)$                       | $\frac{a}{s^2 - a^2}, \quad s >  a $            |
| 7. $\cosh(at)$                       | $\frac{s}{s^2 - a^2}, \quad s >  a $            |
| 8. $u(t-a)$                          | $\frac{e^{-as}}{s}, \quad s > 0$                |
| 9. $u(t-a)f(t-a)$                    | $e^{-as}F(s)$                                   |
| 10. $e^{-at}f(t)$                    | $F(s+a)$  |
| 11. $\int_0^t f(t-\tau)g(\tau)d\tau$ | $F(s)G(s)$                                      |
| 12. $\int_0^t f(\tau)d\tau$          | $\frac{F(s)}{s}$                                |
| 13. $\delta(t-a)$                    | $e^{-as}$                                       |
| 14. $f^{(n)}(t)$                     | $s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$ |

### Variation of parameters

If  $y_1(x)$  and  $y_2(x)$  are two solutions of  $Ly = 0$ , then the particular solution of  $Ly = f(x)$  is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

$$y_1 u_1' + y_2 u_2' = 0,$$

$$y_1' u_1 + y_2' u_2 = f(x).$$