Math 215/255 Final Exam, December 2013

Last Name: ______________________ First Name: ______________________

Student Number: ______________________ Signature: ______________________

Instructions. The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted**. A formula sheet is attached. There are **16 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

Problem #	Value	Grade
1	12	
2	12	
3	16	
4	14	
5	16	
6	18	
7	12	
Total	100	

1. Short answer questions [2 marks each; no partial credit]:

   (a) Write the general solution of $y'' + 2y' + y = 0$.

   (b) Find the *form* (but not the values of the coefficients!) of a particular solution of the nonhomogeneous ODE: $y'' - 4y' + 5y = x\sin(x)$

   (c) $y_1(x) = x$ and $y_2(x) = -x$ are both solutions of the first-order initial value problem: $yy' = x, \quad y(0) = 0$. Does this contradict the existence and uniqueness theorem ('Picard's theorem' or 'Cauchy-Lipshitz theorem')? Explain.

(d) Why are the Laplace transforms of 1 and $u(t)$ (the Heaviside step function) identical?

(e) Write the second-order linear ODE $y'' + ty' + t^2 y = 0$ as a first-order linear system.

(f) Find the Laplace transform $X(s)$ of the solution to the initial value problem: $x''' + x = 0, \quad x(0) = 1,\ x'(0) = 2,\ x''(0) = 3$ (but do **not** compute $x(t)$).

2. (a) [6 marks] Find the solution of

$$y' + x^3 y = 3x^3, \;\; y(0) = 8.$$

(b) [6 marks] Solve

$$\frac{dx}{dt} = \frac{t^2+1}{x^2+1}, \quad x(0) = -1.$$

You may leave your result in implicit form.

3. Suppose the displacement $x(t)$ of a damped mass-spring system subject to sinusoidal forcing of amplitude F_0 is modelled by:

$$x'' + 2x' + 5x = F_0 \sin(2t), \qquad x(0) = 0, \;\; x'(0) = 1.$$

(a) [8 marks] Find and sketch the solution $x(t)$ when $F_0 = 0$ (no forcing).

(b) [8 marks] Now take $F_0 = 1$, and find the steady periodic solution (the part of the solution $x(t)$ which remains as $t \to \infty$).

4. (a) [5 marks] Find the Laplace transform of the following piecewise continuous function by using either the definition of Laplace transform or the second shifting theorem involving a unit step function.

$$g(t) = \begin{cases} 0, & t < 1, \\ e^{(t-1)}, & t \geq 1. \end{cases}$$

(b) [4 marks] Let $Y(s) = \mathcal{L}[y]$ be the Laplace transform of the solution $y(t)$ to the following initial value problem:

$$\begin{cases} y'' - 2y' + y = g(t), \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$$

where $g(t)$ is the function defined in (a). Determine $Y(s)$.

(c) [5 marks] Calculate the inverse Laplace transform of $Y(s)$ obtained in (b) to find the solution, $y(t)$, to the IVP in (b).

5. (a) [8 marks] Find the general solution $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ of the homogeneous linear system

$$\vec{x}' = A\vec{x}, \qquad A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix},$$

and sketch the phase portrait (i.e. some typical trajectories in the $xy-$ plane).

(b) [8 marks] Now find the general solution of the nonhomogenous system (with matrix A as above):

$$\vec{x}' = A\vec{x} + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}.$$

Hint: to find a particular solution, you may use either the method of variation of parameters, or the method of undetermined coefficients; for the latter, try $\vec{x}_p(t) = \begin{bmatrix} (at+b)e^{3t} \\ (ct+d)e^{3t} \end{bmatrix}$.

6. Suppose the interaction between two species ($x = x(t)$ is the "prey" while $y = y(t)$ is the predator) can be modelled by the autonomous system:

$$\begin{cases} \frac{dx}{dt} = x\,(3 - x - y) \\ \frac{dy}{dt} = y\,(1 + x - y) \end{cases}$$

(a) [10 marks] Determine all the critical points and classify their type and stability.

(b) [6 marks] Sketch a phase portrait for the nonlinear system by first sketching a few trajectories near each critical point.

(c) [2 marks] If $x(0) = y(0) = 1$, determine $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

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7. Consider the following initial value problem for a first-order autonomous ODE:

$$\frac{dy}{dt} = -y(y-1)^2, \quad y(0) = y_0.$$

(a) [4 marks] Find the equilibrium solutions (critical points), determine the stability (stable or unstable) of each, and find $\lim_{t\to\infty} y(t)$ for all possible values of the initial condition y_0.

(b) [4 marks] If $y_0 = 3/2$, use Euler's method with step size $h = 1$ to approximate the solution $y(2)$ at time $t = 2$.

(c) [4 marks] Again if $y_0 = 3/2$, use Euler's method with the larger step size $h = 2$ to approximate the solution $y(2)$ at time $t = 2$. Predict what will happen if you continue to use this scheme (with $h = 2$) to approximate $y(t)$ for large t. Compare this with your result from part (a), and explain any discrepancies.

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$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$(-t)^n f(t)$	$F^{(n)}(s)$
$f^{'}(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$u(t-a)$	e^{-as}/s
$\delta(t-a)$	e^{-as}
$f(t) * g(t)$	$F(s)G(s)$

Table 1: Tables of Laplace Transforms