Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - April 2012

Mathematics 215

H. Dixit, M. Ward

Closed book examination. No notes, texts, or calculators allowed. Time: $2\frac{1}{2}$ hours

Marks

[15] **1.** Two First Order Differential Equations:

(i) Determine the explicit solution $y(t)$ for $t \geq 0$ to

$$y' - 2y = t^2 e^{2t}, \qquad y(0) = 1.$$

(ii) Determine the explicit solution $y(t)$ for $t \geq 0$ to

$$y' = 2y^2 + ty^2, \qquad y(0) = 1.$$

In addition, for (ii) determine the interval $0 \leq t < T$ of existence of your solution, and give a qualitatively accurate plot of the solution y versus t on this interval.

[10] **2.** Consider population growth under a logistic growth law but subject to a constant harvesting rate, which is modeled by the solution $y(t)$ to

$$\frac{dy}{dt} = f(y), \quad \text{where} \quad f(y) = y(4 - y) - 3.$$

(i) Determine the two equilibrium solutions and plot $f(y)$ versus y on $y \geq 0$.

(ii) Plot y versus t for various initial conditions $y(0) = y_0$ with $y_0 \geq 0$.

(iii) Suppose that $y(0) = 1/2$. Determine the time T for which the population becomes extinct, i.e. $y(T) = 0$. (Hint: it is easy to explicitly calculate the integral that determines T).

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[15] **3.** Let $\omega > 0$ be a constant, and consider the initial value problem for $y = y(t)$

$$y'' + 5y = \cos(\omega t)\,, \quad \text{with} \quad y(0) = 0\,, \quad y'(0) = 0\,.$$

(i) Find the explicit solution to this problem. Account for all $\omega > 0$ (Hint: there are two cases of ω to consider).

(ii) If $\omega = \sqrt{5}$ what would be the effect on the solution of adding a little damping of the form cy' for some $c > 0$ small? Identify at least two key qualitative changes in comparison with the solution in (i).

[15] **4.** Consider the second order differential equation

$$t^2 y'' + ty' - y = t \ln t\,.$$

(i) Find the general solution to this problem in an explicit a form as you can (Hint: you are given that $y = t$ is a solution to the homogeneous problem $t^2y'' + ty' - y = 0$).

(ii) Consider the initial conditions $y(1) = a$ and $y'(1) = 0$ where a is a parameter. Find the value of the parameter a for which $\lim_{t\to 0} y(t) = 0$.

[15] **5.** Let $T > 0$ be a constant, and define the piecewise continuous function $f(t)$ by

$$f(t) = \begin{cases} t\,, & 0 \le t < T, \\ 0\,, & t \ge T. \end{cases}$$

(i) Calculate $F(s)$, where $F(s)$ is the Laplace transform of $f(t)$.

(ii) With $f(t)$ as given above, calculate in an explicit a form as you can the solution $y(t)$ to

$$y'' + 3y' + 2y = f(t) \quad \text{with} \quad y(0) = 1\,, \quad y'(0) = 0\,.$$

(iii) Give a rough sketch of the solution y versus t when $T = 10$. For this value of T, give an approximate expression for the solution $y(t)$ when $t = 9$.

[15] **6.** Consider the inhomogeneous linear system for $\mathbf{x}(t)$ given by

$$\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^{-t} \\ 1 \end{pmatrix}\,, \quad \text{where} \quad A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\,.$$

(i) Calculate the general solution for this problem.

(ii) Calculate the vector $\mathbf{b}$ for which $\lim_{t\to\infty} \mathbf{x}' = \mathbf{b}$ for any initial condition $\mathbf{x}(0)$.

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[15] **7.** Consider the following nonlinear system for $x = x(t)$ and $y = y(t)$:

$$\frac{dx}{dt} = x(3 - x - 2y)\,, \qquad \frac{dy}{dt} = y(2 - x - y)\,.$$

(i) Find the equilibrium points and classify their type and stability (i.e. saddle point, node, spiral point etc..) based on the Jacobian matrix associated with the linearization of the nonlinear system.

(ii)) Plot the solution trajectories **near** each of the equilibrium points.

(iii) By considering the nullclines of the system, give a qualitatively accurate sketch of the phase plane for this system in the entire first quadrant $x \geq 0$, $y \geq 0$.

(iv) Let $x(0) = 1$ and $y(0) = 1/2$. For this initial condition, calculate $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$.

[100] **Total Marks**

The End

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19.	$(-1)^n f(t)$	$F^{(n)}(s)$