Be sure this exam has 10 pages including the cover

The University of British Columbia

Final Exam – December 2007
Mathematics 215/255, Ordinary Differential Equations

Name ______________________ **Signature** ______________________
Student Number ______________________
Circle Section: **101** Phan **102** Burghelea **103** Tsai **104** Flowers **105** Brydges

This exam consists of **8** questions worth **100** marks in total. No aids are permitted.

Problem	max score	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	15	
6.	15	
7.	15	
8.	15	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

(10 points) 1. A bath contains water. Let the depth of water at time t be $y = y(t)$ ins. At time $t = 0$ the plug is pulled and water exits via the drain at the rate of $2y$ inches per minute.

(a) The bath is filled to an initial depth of 10 inches. Find the depth y as a function of time.

(b) Same problem as in the previous part except that at time $t = 0$ the tap is turned on and water enters the bath at rate 3 inches per minute.

(10 points) 2. Consider

$$t + ye^{2ty} + ate^{2ty}\frac{dy}{dt} = 0. \tag{1}$$

(a) Determine a such that (1) is exact.

(b) Solve (1) when a has the value found in the previous part.

(10 points) 3. For the equation $y' = 1 - x - y$ with initial condition $y(0) = 2$ write down the Euler recursion with $h = 0.1$ for finding the solution approximately and use it to approximate $y(0.2)$.

(10 points) 4. (a) Find the general solution to $y'' - 2y' + 2y = 0$.

(b) Find a particular solution to $y'' - 2y' + 2y = e^t \cos t$

(c) Find the general solution to $y'' - 2y' + 2y = e^t \cos t$.

(15 points) 5. (a) Solve $y'' + y' = \begin{cases} t & \text{if } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$ when $y(0) = y'(0) = 0$.

(b) Find $F(s) = \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$, given that $\mathcal{L}\{\sqrt{t}\} = \frac{1}{2}\sqrt{\pi}s^{-3/2}$ and $F(1) = \sqrt{\pi}$.

(c) Find the inverse Laplace transform of $\dfrac{e^{-2s}}{(s+1)^2}$.

(15 points) 6. Consider

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}. \tag{2}$$

(a) Find the solution of (2) such that $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Find the solution of (2) such that $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Find the matrix e^{tA}.

(d) Draw the phase portrait for (2) including directions of flow.

(15 points) 7. Consider $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y + y^3 \\ 2x + 1 \end{bmatrix}$

(a) Find the orbits.

(b) Find and classify equilibrium point(s).

(c) Draw the local phase plane portrait near the equilibrium point(s).

(15 points) 8. For the following questions, fill in the answers in the boxes. No work need to be shown and no partial credit will be given. **Anything written outside of the boxes is ignored.**

(a) Convert the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \qquad y(0) = 1, \quad y'(0) = 2.$$

into an equivalent first order system of the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}.$$

Answer =

(b) In the method of Picard iteration the equation $\frac{dy}{dt} = y^2$, with initial condition $y(0) = 7$ is solved by a recursion of the form

$$y_{n+1}(t) = ? + \int_?^? ?\, ds, \qquad y_0(t) = ?$$

Find $y_1(t)$.

Answer =

(c) Is the solution to $\frac{dy}{dt} = y^{2/3}$ with $y(0) = 1$ unique?

Answer =

(d) Is the solution to $\frac{dy}{dt} = y^{2/3}$ with $y(0) = 0$ unique?

Answer =

(e) The Wronskian of two solutions of $y'' + 3y' + 2y = 0$ is proportional to e^{at}. Find a.

Answer =

Table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
C	$\frac{C}{s}$
e^{at}	$\frac{1}{s-a}, s > a$
$\sin(at)$	$\frac{a}{s^2+a^2},\ s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2},\ s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2},\ s > \lvert a\rvert$
$\cosh(at)$	$\frac{s}{s^2-a^2},\ s > \lvert a\rvert$
$\delta(t-c)$	e^{-cs}
$H_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
$H_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct} \cdot f(t)$	$F(s-c)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$