

The University of British Columbia

Final Examination - April 10, 2013

Mathematics 210

Spring, 2013

Closed book examination

Time: 2.5 hours

**Special Instructions:**

No books, notes, or calculators are allowed.

**Rules governing examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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|--------|--|----|
| Part A |  | 30 |
| B1     |  | 5  |
| B2     |  | 5  |
| B3     |  | 5  |
| B4     |  | 5  |
| Total  |  | 50 |

**Instructions:**

- Do all fifteen questions in part A, each worth 2 points.
- Do all four questions in part B, each worth 5 points.
- total 50 marks.
- Justify all answers.
- Write your answers in booklets.

**Part A: do all fifteen questions, each worth 2 marks**

**A1.** [2marks] What is the output of the following MATLAB commands?

```
A = zeros(2,3);
A(1,:) = 1:3;
A(2,1) = 6;
A
```

**A2.** [2] Calculate the Jacobian matrix of the system

$$\begin{aligned} f_1(x_1, x_2) &= \sin(x_1 x_2) \\ f_2(x_1, x_2) &= e^{x_1 x_2} - x_1^2 \end{aligned}$$

**A3.** [2] Consider the scalar, autonomous DE below

$$\frac{du}{dt} = u^2 - u^3$$

with  $u(0) = 2$ . Approximate  $u(0.2)$  using two steps of Euler's method with time step  $k = 0.1$ . It is not necessary to simplify your answer.

**A4.** [2] Consider the function

$$f(x) = 1 - 3x + x^3$$

which is known to have a root  $x \in [0, 1]$ . What is the estimate of the root obtained after one Newton step beginning at initial guess  $x = 1/2$ ?

**A5.** [2] Consider the function

$$f(x) = 1 - 3x + x^3$$

which is known to have a root  $x \in [0, 1]$ . Beginning with this interval, do two steps of the bisection method. After these steps, what interval is guaranteed to have the root?

- A6.** [2] Bob is writing a MATLAB code that is supposed to implement Newton's method to find a root of

$$f(x) = \cos(x) - x^2.$$

His script is given below:

```
tol = 1e-6;
res = 2*tol;
while res > tol
    x = x-(cos(x)-x^2)/(-sin(x)-2*x);
end
```

Bon is frustrated because his script never ends. Add an additional line to Bob's script so that it finishes properly.

- A7.** [2] Bob is writing a MATLAB function to input to `ode45` to solve the DE system below:

$$\begin{aligned}\frac{dx}{dt} &= x^3y + x \\ \frac{dy}{dt} &= x^2 + y^2.\end{aligned}$$

His function file `Bobf.m` is given below:

```
function rhs = Bobf(xy);
x = xy(1);
y = xy(2);
rhs(1) = x^3*y+x;
rhs(2) = x^2 + y^2;
end
```

What is the output to the command `Bobf([2 3]')`?

- A8.** [2] When Bob inputs the function file in the question above into `ode45`, it does not behave correctly. What two modifications should be made to his function?

- A9.** [2] Bob is trying to use Maple to solve a scalar autonomous DE with a right hand side function  $f(u)$  that he has previously defined. He begins to enter the commands to do this with the line

```
de = diff(u(t),t) = f(u(t));
```

but gets the result `false` from this command. Help Bob by telling him how to fix this line so that it will function properly.

**A10.** [2] The scalar autonomous DE

$$\frac{dx}{dt} = f(x)$$

is known to have an equilibrium point at  $x = 2$ . The quadratic Taylor polynomial approximation to  $f(x)$  based at  $x = 2$  is given below:

$$T_2(x) = -\frac{1}{2}(x - 2) + \frac{1}{6}(x - 2)^2.$$

Is the fixed point stable? Justify briefly.

**A11.** [2] The scalar iterative map

$$x^{(n+1)} = f(x^{(n)})$$

is known to have a fixed point at  $x = 1$ . The quadratic Taylor polynomial approximation to  $f(x)$  based at  $x = 1$  is given below:

$$T_2(x) = 1 + (x - 1) + \frac{1}{4}(x - 1)^2.$$

Is the fixed point stable? Justify briefly.

**A12.** [2] Euler's method is used on a DE problem from initial conditions at  $t = 0$  to an approximate value at  $t = 1$ . The exact solution to the problem is known and the error in the approximation when  $N = 100$  time steps of size  $k = 0.01$  are used is  $4.2317e - 4$ . What would you expect the error to be (approximately) if  $N = 200$  steps of size  $k = 0.005$  were used? Justify briefly.

**A13.** [2] Show that

$$\left(1 + \frac{x}{n}\right)^n \leq e^x$$

for all  $x > 0$  and all positive integers  $n$ .

**A14.** [2] Consider applying Euler's method to approximate  $u(t)$  that satisfies

$$\frac{du}{dt} = -10u$$

with  $u(0) = 1$ . The exact solution is  $u(t) = e^{-10t}$ . What values of the time step  $k$  for Euler's method match the solution for large time, that is satisfy

$$\lim_{n \rightarrow \infty} U^{(n)} = 0,$$

where  $U^{(n)}$  is the Euler approximation of  $u(nk)$ ?

**A15.** [2] The eigenanalysis of a random walk transition matrix  $P$  is performed using the MATLAB command  $[T D] = \text{eig}(P)$  which gives the following results:

$$T = \begin{bmatrix} 0.6666 & 0.7071 & 0.7071 \\ 0.6666 & 0.0000 & -0.7071 \\ 0.3333 & -0.7071 & 0.0000 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 0.0000 & 0.3333 \end{bmatrix}$$

From this information, determine

$$\lim_{n \rightarrow \infty} P^n [1, 0, 0]^T.$$

**Part B: do all four questions, each worth 5 points**

**B1.** A random walk transition matrix  $P$  is given below

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1/4 \\ 1/3 & 1/6 & 1/4 \end{bmatrix}$$

Some MATLAB computations with  $P$  determine that

$$P^4 \approx \begin{bmatrix} 0.3754 & 0.3744 & 0.3753 \\ 0.3744 & 0.3763 & 0.3740 \\ 0.2502 & 0.2493 & 0.2507 \end{bmatrix} \quad \text{and} \quad P^{25} \approx \begin{bmatrix} 0.3750 & 0.3750 & 0.3750 \\ 0.3750 & 0.3750 & 0.3750 \\ 0.2500 & 0.2500 & 0.2500 \end{bmatrix}$$

- (a) [2] If the system begins (time 0) in state 1, how likely is it that it will be in state 3 at time 1?
- (b) [1] If the system begins in state 1, what state is it most likely to be in at time 4?
- (c) [1] What is your estimate of the probability of being in state 1 after many iterations no matter where you start from?
- (d) [1] What MATLAB computation would you do to be more confident in your answer to part (c) above?

**B2.** Consider the second order DE for  $u(t)$  given below:

$$\ddot{u} + u^5 = 0.$$

- (a) [3] Multiply the equation above by  $\dot{u}$  and identify a quantity  $E(u, \dot{u})$  that is constant for solutions.
- (b) [2] Using your result above, determine whether the solution with initial data

$$u(0) = 1, \quad \dot{u}(0) = 2$$

remains bounded or has

$$\lim_{t \rightarrow \infty} |u(t)| = \infty$$

**B3.** Consider the vector iterative map given below

$$\begin{aligned} x^{(n+1)} &= f(x^{(n)}, y^{(n)}) \\ y^{(n+1)} &= g(x^{(n)}, y^{(n)}) \end{aligned}$$

with

$$\begin{aligned} f(x, y) &= x^2 - y^2 + xy + 2 \\ g(x, y) &= x^2 y^2 - 2 \end{aligned}$$

- (a) [2] Find the Jacobian matrix for the map.
- (b) [1] Note that the map has a fixed point  $(1, 2)$ . Evaluate the Jacobian at this point.
- (c) [1] Find the eigenvalues of the Jacobian above.
- (d) [1] Determine whether  $(1, 2)$  is a stable or unstable sized point from your answer to (c) above.

**B4.** You are going to write a simulation for waiting times in lines at a large grocery store. The simulation will update the state of the lines every 10 seconds. The aim of the simulation is to estimate the distribution of customer waiting times. At each time, the state of the lines will be recorded in a data structure.

- (a) [3] One element of the data structure will be  $N$ , the number of cashiers (lines). What other elements would you have in your data structure? Give the data types of the elements you propose (integers, arrays with given lengths, etc.).
- (b) [2] Describe two things that you would want to have measured experimentally before writing your simulation.