The University of British Columbia

Final Examination - April 11, 2017

Mathematics 152

All Sections

Closed book examination. No calculators. Time: 2.5 hours

Family Name ______________________ **Given Name** ______________________

Student Number ______________________ **Signature** ______________________

Section: ______________________ **Instructor:** ______________________

Special Instructions:

No books, notes, calculators or any electronic devices are allowed. Show all your work. In part B questions, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

For Questions A1-A3 below, let

$$u = 1 + 3i \quad \text{and} \quad z = 2 - i$$

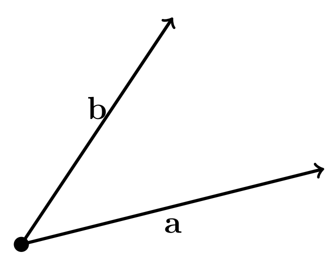
Your answers should be in the form $x + iy$ with x and y determined.

A1: Compute $2u - 3z$.

A2: Compute uz.

A3: Compute u/z.

A4: Draw the vector $\mathbf{a} - \mathbf{b}$ on the picture below.



A5: Give a parametric equation for the line in $\mathbb{R}^3$ passing through the points $(0, 5, 1)$ and $(2, 2, 13)$.

A6: Find a normal vector to the plane containing the points $(3, 0, 1)$, $(4, 1, 2)$, and $(4, -2, 4)$.

A7: let $\mathcal{E}$ be a system of linear equations, and let $\mathcal{E}_0$ be the associated homogeneous system of linear equations. If $[3, 1, 5, 8]$ is a solution to $\mathcal{E}$, and the solutions to $\mathcal{E}_0$ are $s[2, 3, 0, 1]$, give all solutions to $\mathcal{E}$.

For questions A8 and A9 below, let $\mathbf{v} = [1, 2, 3, 4]$ and $\mathbf{x} = [a, a, a, a]$ for some constant a. Briefly justify your answers.

A8: For which nonzero value or values of a (if any) is $|\mathbf{v}| = |\mathbf{x}|$?

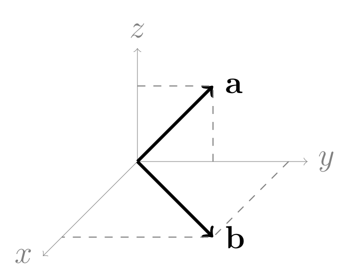
A9: For which nonzero value or values of a (if any) is $\mathbf{v}$ perpendicular to $\mathbf{x}$?

A10: Give the intersection of the planes $x + y - 2z = 0$ and $2x - 5y + 7z = 0$ in parametric form.

A11: Let $\mathbf{a} = [0, 1, 1]$ and $\mathbf{b} = [2, 2, 0]$. Sketch the set of points

$$\{s\mathbf{a} + (1 - s)\mathbf{b} : 0 \leq s \leq 1\}$$

on the graph below.



A12: What is the output after the following lines of MATLAB code?

```
A = [1 2 3; 4 5 6; 7 8 9];
A(2,3)
```

A13: Circle each of the following statements that is true for *all* invertible $n \times n$ matrices A:

(a) The rank of A is n.

(b) $\det A = 0$.

(c) The reduced row echelon form of A is the $n \times n$ identity matrix.

(d) A^{-1} exists.

(e) Columns of A are linearly independent.

(f) $A = A^T$.

A14: Circle *all* possible types of solution sets that a system of 3 equations in six unknowns can have:

(a) No solutions

(b) A unique solution

(c) An infinite number of solutions

(d) A two-dimensional set of solutions

(e) A four-dimensional set of solutions

A15: Write down the matrix A that will result from the following lines of MATLAB code:

```
A=zeros(3,4);
A(3,:) = [1 2 3 4];
for i=1:3
    A(i,i+1) = 5;
end
```

A16: For which value or values of θ in the interval $(-\pi/2, \pi/2]$ does the matrix Ref_θ have

$$\begin{bmatrix} \sqrt{3} \\ 3 \end{bmatrix}$$

as an eigenvector? *Recall:* Ref_θ is reflection through a line that makes an angle θ with the x-axis.

A17: Compute the eigenvalues of the matrix below.

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

A18: Find a parametric form of the equation of the plane $x - 3y + 2z = 5$ in $\mathbb{R}^3$.

A19: Suppose λ is an eigenvalue of an invertible matrix A with eigenvector $\mathbf{x}$. Give an eigenvalue-eigenvector pair of A^{-1}.

A20: Suppose `P` was the transition matrix for a random walk with 8 states that had been entered into MATLAB. What commands would you use to compute the probability that if the system started in state 1, it would be in state 7 after 13 time steps?

A21: $A = \begin{bmatrix} a & \frac{1}{2} \\ -\frac{1}{2} & b \end{bmatrix}$, find values a and b such that $A^{-1} = A^T$. *Note:* one set of such values is sufficient.

A22 Describe all cubic polynomials $y(x) = ax^3 + bx^2 + cx + d$ passing through the points $(0, 1)$, $(1, 2)$, and $(-1, 4)$.

A23: Let A and B be matrices and let $\mathbf{x}$ be a vector. If $(A\mathbf{x})^T B$ is an 4×7 matrix, is $\mathbf{x}$ a row vector or a column vector, and how many entries does it have? Justify your answer.

A24: Solve the following system of equations.

$$\begin{array}{rcrcl} 2x & + & 3y & = & 2+17i \\ -x & + & y & = & 1+4i \end{array}$$

Your answer should be a vector with each component a complex number of the form $a+ib$ with a and b determined.

A25: A matrix A is entered into MATLAB. The eigenanalysis of A is performed using the command `[T D] = eig(A)` which gives the following results:

```
T =
    0.5257    0.0995
    0.8507    0.9950
D =
    1.0000         0
         0    3.0000
```

Using these results, determine $A^3[1,10]^T$.

A26: Solve the matrix equation $AX+B=C$ for matrix X, where
$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

A27: The matrix A represents a projection in $\mathbb{R}^2$. It is known that $\lambda_1 = 0$ is one eigenvalue with corresponding eigenvector $\mathbf{v}_1 = [-3\ 1]^T$. Find the matrix A.

A28: Let P be the plane in $\mathbb{R}^3$ with equation $x+y-z=0$. Find two more planes Q and R such that all three planes are different and the intersection is the line $s[3,-2,1]+[2,2,4]$.

A29: Give a vector in $\mathbb{R}^3$ that makes an angle of $\frac{\pi}{4}$ radians with the vector $\mathbf{v} = [3,1,5]$. Describe briefly the procedure you used to get your answer.

A30: Let P be a plane and L a line in $\mathbb{R}^3$ given by equations

$$P : x+y+z = 1$$

$$L : u\begin{bmatrix}1\\1\\1\end{bmatrix} \quad u \in \mathbb{R}.$$

Find *all* linear transformations such that the image of P is L (that is the set of all outputs is the line L when all points on the plane P are taken as inputs).

Part B - Long Answer Questions, 5 marks each

B1: Three engineering students, Xiuying, Arjan, and Marianne decide to take a summer job painting homes. Xiuying paints three times as fast as Marianne and twice as fast as Arjan and Marianne combined. All three together paint a room in four hours.

(a) [2 marks] Let $\mathbf{x} = (x_1, x_2, x_3)^T$ be the vector, where x_1, x_2 and x_3 are respectively the number of rooms that Xiuying, Arjan, and Marianne can paint in an hour. Describe the information above as a linear system in the form

$$A\mathbf{x} = \mathbf{b}$$

(write A and $\mathbf{b}$ with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. How many rooms can each of the three friends paint in an hour? Check that your answer matches the original information in the question.

B2: For each matrix below left, interpret the matrix as a probability transition matrix, and match it to its corresponding equilibrium probability on the right. One mark each. *Note* that answers in the list (I)-(VII) may be a match more than once or not at all.

$(A) \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$ $\qquad (I) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(B) \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $\qquad (II) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(C) \begin{bmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{bmatrix}$ $\qquad (III) \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$(D) \begin{bmatrix} 9/10 & 0 \\ 1/10 & 1 \end{bmatrix}$ $\qquad (IV) \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

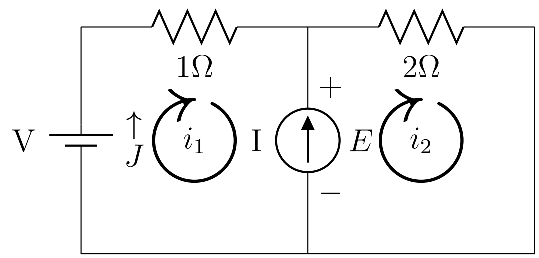
$(E) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad (V) \begin{bmatrix} \frac{9}{10} \\ \frac{1}{10} \end{bmatrix}$

(VI) not a probability transition matrix

(VII) equilibrium probability not in the list

(VII) no equilibrium probability

B3: Consider the resistor network below.



(a) [2 marks] Using the loop current technique as described in the online notes and computer labs, write three linear equations that can be solved for i_1, i_2, and E. Let the sources I and V be parameters in your equations.

(b) [2] J is the current through the voltage source (note that $J = i_1$). Write J and E in terms of I and V.

(c) [1] Suppose $J = 10A$ when $I = 5$A. What is the voltage drop V at the voltage source?

B4: A model of love affairs is described by the following system of differential equations

$$R' = R + 4J,$$
$$J' = -2R - 3J,$$

where $R(t)$ and $J(t)$ are, respectively, Romeo's and Juliet's amount of love for each other at time t.

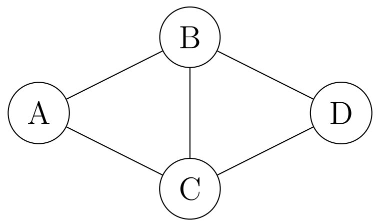
(a) [1 mark] Express the system in matrix form $d\mathbf{x}/dt = A\mathbf{x}$, where $\mathbf{x}(t) = [R(t)\ J(t)]^T$.

(b) [1] The eigenvalues of A are $\lambda = -1 \pm 2i$. Find the eigenvectors of the matrix with these eigenvalues.

(c) [1] Express the general solution to the system in either real or complex form.

(d) [2] Assuming that initially the two have equal love for each other, i.e. $\mathbf{x}(0) = [1\ 1]^T$, find $\mathbf{x}(t)$ in its real form (involving no complex numbers).

B5: Suppose four towns are connected by roads in the configuration shown below. A random driver wakes up every morning and flips a coin. If the coin is heads, she stays where she is for the day. If the coin is tails, she drives to the next town, choosing one of the roads with no preference. (For example, if she leaves Town A, she is equally likely to go to Town B or Town C, but she will not go to Town D that day.)



(a) [2 marks] Write the probability transition matrix P for the random driver. Use the ordering A, B, C, D.

(b) [1] If the driver starts in Town A, what is the probability she will be in Town A two days later?

(c) [2] What is the equilibrium probability of the system?

B6: The matrix A below represents a transformation that reflects vectors in $\mathbb{R}^3$ in a plane P that contains the origin.

$$A = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

Note: you can use geometric insight to greatly simplify the calculations below.

(a) [2 marks] Find all eigenvalues and a basis of eigenvectors of A.

(b) [2] Find an equation for P (equation form).

(c) [1] Find A^n for all integers $n \geq 1$.