The University of British Columbia

Final Examination - April 16, 2014

Mathematics 152

All Sections

Closed book examination. No calculators.   Time: 2.5 hours

Last Name _____ **First** _____ **Signature** _____

Student Number _____

Section : _____

Instructor : _____

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

A1: Compute
$$(2 + 3i)/(4 - 2i).$$

Your answer should be in the form $a + bi$ where a and b are real numbers to be determined.

A2: For what value or values of a (if any) is the matrix below invertible?

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & a & 2 \end{bmatrix}$$

A3: Circle the *one* correct answer below. A linear system of three equations in five unknowns has

(a) always a unique solution.

(b) either a unique solution or no solutions.

(c) either a unique solution or an infinite number of solutions.

(d) either no solutions or an infinite number of solutions.

(e) either a unique solution or exactly two distinct solutions.

A4: Consider a random walk with three states. A walker that begins at state 1 always goes to state 2; always moves to state 3 when beginning in state 2; and is equally likely to be in any state after starting in state 3. Write the probability transition matrix for this random walk.

A5: What is output after the following lines of MATLAB code?

```
A = [1 2 3 4 5; 6 7 8 9 10];
A(:,2)
```

A6: Let $\mathbf{A}$ be a 2×2 matrix with real entries. Suppose that

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$ with eigenvalue $2 + 3i$. What is another eigenvalue of $\mathbf{A}$ and its associated eigenvector?

A7: Find an eigenvector of eigenvalue 2 for the matrix

$$\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

A8: Circle *all* the statements below that *must be true* for any 3×3 matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. It is known that $\det \mathbf{A} = 4$, $\det \mathbf{B} = 3$ and $\det \mathbf{C} = 0$.

(a) $\det(2\mathbf{A}) = 8$

(b) 0 is an eigenvalue of $\mathbf{C}$.

(c) $\det(\mathbf{A} + \mathbf{B}) = 7$.

(d) $\mathbf{ABC}$ is invertible.

(e) $\det \mathbf{B}^T = 3$.

A9: Compute the inverse of

$$\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

Questions **A10** and **A11** below concern the line L through the origin with direction $[3, -4]$.

A10: Find the projection of the vector
$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
onto L.

A11: Find the matrix that represents reflection across the line L.

A12: Compute the determinant of

$$\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 0 & 2 & 3 \\ -2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 2 \end{bmatrix}$$

A13: Compute the determinant of **AB** where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 9 & 1 & 1 \end{bmatrix}$$

Problems **A14** and **A15** below concern solving systems of linear equations with MATLAB. For both questions, a system is entered into MATLAB as an augmented matrix `A`. In the two questions below, the result of `rref(A)` is shown. Use the information to determine whether each system has a solution or solutions. Either state that there is no solution and justify briefly or write down the solution or parametric form of all solutions.

A14:

1	0	3	0
0	1	0	0
0	0	0	1

A15:

1	0	1	2
0	1	1	1

For questions **A16-A17** below, consider the system with parameters a and b:

$$2x + y = a$$
$$bx - 2y = 6$$

A16: For what values of a and b (if any) does the system have a unique solution?

A17: For what values of a and b (if any) does the system have infinitely many solutions?
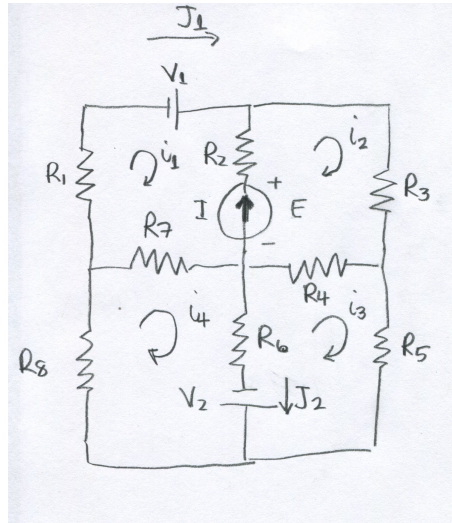
A18: Consider the differential equation system for $\mathbf{x}(t)$

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x}$$

Write the general solution to the system.

A19: What is the shortest distance between points on the parallel lines?

$$\begin{aligned} 2x - y &= 2 \\ 4x - 2y &= 8 \end{aligned}$$

A20: Consider the resistor network above. Using the loop current method you have learned in the lectures and labs, write the equation for voltages around loop 1. Your equation should have the resistances and sources as parameters.

A21: Consider the resistor network above. Using the loop current method, write the equation that matches the loop currents to the current source.

A22: Consider the resistor network above. Using MATLAB for a certain set of resistance values, it is found that

$$
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ E \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_1 \\ 0 \\ -V_2 \\ V_2 \\ I \end{bmatrix}, \text{ where } \mathbf{A} = \begin{bmatrix} 7/27 & 7/27 & 10/81 & 4/81 & -43/81 \\ 7/27 & 7/27 & 10/81 & 4/81 & 38/81 \\ 4/27 & 4/27 & 25/81 & 10/81 & 14/81 \\ 1/9 & 1/9 & 4/27 & 7/27 & -1/27 \\ 13/27 & -14/27 & -20/81 & -8/81 & 248/81 \end{bmatrix}
$$

From this result, determine the *second row* of the matrix **B** such that

$$
\begin{bmatrix} J_1 \\ J_2 \\ E \end{bmatrix} = \mathbf{B} \begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}
$$

A23: A 2×2 matrix $\mathbf{A}$ is known to have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the general solution to $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$.

A24: For the same system as in **A23** above, find the solution with initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A25: A 2×2 matrix $\mathbf{A}$ is known to have eigenvalue $\lambda_1 = 1+i$ with corresponding eigenvector

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Find the general solution to $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$.

A26: For the same system as in **A25** above, find the solution with initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Your answer should be in real form (involving no complex quantities).

A27: What is the polar representation of $z = 1 + i$?

A28: What is the root of

$$z^5 = 3$$

that lies in the second quadrant of the complex plane? *Note:* the second quadrant has complex numbers with negative real parts and positive imaginary parts.

A29: Find the inverse of

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

A30: You are using MATLAB to investigate a large random walk (15 locations). You spend a long time entering the transition matrix P correctly. You looked at the eigenvalues of P and found they were all positive. You computed `A=P^3` and `B=P^4`. You then realized that you have over-written the matrix P but `A` and `B` are still saved. Describe briefly how you could recover P and not have to type it in all over again.

Part B - Long Answer Questions, 5 marks each

B1: Let $\mathbf{A}$ be a 2×2 matrix in the form

$$\begin{bmatrix} 2 & 4 \\ 1 & a \end{bmatrix}$$

    (a) [1 mark] Find a such that $\mathbf{A}$ is not invertible.

    (b) [1] For the value of a above, find the eigenvalues of $\mathbf{A}$.

    (c) [2] Find the eigenvectors associated with the eigenvalues found above.

    (d) [1] Do the eigenvectors found above form a basis of $\mathbb{R}^2$? Justify briefly.

B2: Let d be a constant to be determined. It is known that the linear system in augmented matrix form

$$
\left[\begin{array}{ccccc|c}
1 & 2 & -1 & 0 & 4 & 1 \\
2 & 3 & -1 & -1 & 5 & 0 \\
0 & 1 & -1 & 1 & 3 & 2 \\
3 & 5 & -2 & 0 & 10 & d
\end{array}\right]
\sim
\left[\begin{array}{ccccc|c}
1 & 1 & 0 & 1 & 3 & 3 \\
0 & 1 & -1 & -3 & -1 & -6 \\
0 & 0 & 0 & 2 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

where $\sim$ denotes allowable operations to put the augmented matrix into row echelon form.

(a) [2 marks] Find the reduced row echelon form of the given system.

(b) [2] Find the solution or solutions of the system (if any).

(c) [1] What is the value of d?

B3: Let

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

be the transition matrix of a random walk. It is known that $\mathbf{P}$ has three distinct eigenvalues. Two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 0.2$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

(a) [2 marks] Find the third eigenvalue λ_3 and its associated eigenvector $\mathbf{v}_3$.

(b) [2] If the initial state is

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}_{10} = P^{10}\mathbf{x}_0$. *Hint:* $\mathbf{x}_0$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

(c) [1] What is $\lim_{n \to \infty} x_n$?

B4: Let T be a linear transformation. Suppose that

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\-1\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\0\end{bmatrix}.$$

(a) [1 mark] Calculate $T(\mathbf{e}_1)$, where $\mathbf{e}_1 = [1\ 0\ 0]^T$ is the unit vector in x_1-direction of $\mathbb{R}^3$.

(b) [2] Find the matrix representation of T.

(c) [2] Calculate $T^{20}(\mathbf{e}_1)$.

B5: Let P be the plane with equation $x_1 + x_2 + x_3 = 0$.

  (a) [2 marks] Give a parametric form of the line L through an arbitrary point (a, b, c) which intersects P in a perpendicular direction.

  (b) [2] The projection onto the plane P is a linear transformation. Find its matrix representation $\mathbf{A}$. *Hint:* find the coordinates of the intersection of the line L above and P.

  (c) [1] Explain briefly why 0 and 1 are eigenvalues of $\mathbf{A}$. Determine all eigenvectors of $\mathbf{A}$. Do both using a geometric argument.

B6: Let $\mathbf{v_1} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, $\quad \mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix}$.

(a) [1 mark] For what value of a are the three vectors $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ listed above linearly dependent?

(b) [2] For the value of a found in (a), find a linear relation between the three vectors, that is find scalars α, β, γ such that $\alpha \mathbf{v_1} + \beta \mathbf{v_2} + \gamma \mathbf{v_3} = \mathbf{0}$. Express the result as a vector $\mathbf{k} = [\alpha \ \beta \ \gamma]^T$.

(c) [1] For the value of a found in (a), let $A = [\mathbf{v_1} \ \mathbf{v_2} \ \mathbf{v_3}]$ and $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$. For what value of b does $A\mathbf{x} = \mathbf{b}$ have a solution?

(d) [1] For the values of a, b found in (a) and (c), solve $A\mathbf{x} = \mathbf{b}$. Explain the relation between this solution and the result in (b).

The End