The University of British Columbia

Final Examination - April 15, 2013

Mathematics 152 (ALL SECTIONS)

Closed book examination Time: 2.5 hours

Last Name ____________________ **First** ______________ **Signature** ____________________

Student Number ____________________ **Section** ____________________

Special Instructions:

No books, notes, or calculators are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Points	Score
Part A	60	
B1	10	
B2	10	
B3	10	
B4	10	
B5	10	
B6	10	
Total	120	

Part A: Short Answer Problems

In the next 3 problems, let $\mathbf{a} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ -4 \\ 12 \end{bmatrix}$.

(A1) (2 points) What is $\mathbf{a} \times \mathbf{b}$?

(A2) (2 points) What is $\mathrm{Proj}_{\mathbf{c}}\mathbf{a}$?

(A3) (2 points) Let $\mathbf{d} = \begin{bmatrix} -1 \\ k \\ 2k+1 \end{bmatrix}$ where k is a parameter. Find the value of k for which $\mathbf{a}$ and $\mathbf{d}$ are orthogonal.

In the next 2 problems, consider the matrix M given by

$$\begin{bmatrix} 1 & 7 & 3 \\ 0 & p & 5 \\ 1 & pq+7 & 8 \end{bmatrix}$$

(A4) (2 points) For what p and q does the matrix M have rank 2?

(A5) (2 points) Choose such p and q (so that M has rank 2). Give an example of a vector $\mathbf{b}$ for which the linear system $M\mathbf{x} = \mathbf{b}$ has no solution.

(A6) (2 points) Is it possible for three vectors in $\mathbb{R}^2$ to be linearly independent? Briefly justify your answer.

(A7) (2 points) Is it possible for the intersection between two planes in $\mathbb{R}^3$ with non-parallel normal vectors to contain two linearly independent vectors?

(A8) (2 points) Give **all solutions** to the homogeneous system $A\mathbf{x} = \mathbf{0}$,where

$$A = \begin{bmatrix} 1 & 7 & 7 & \dots & 7 \\ 0 & 2 & 7 & \dots & 7 \\ 0 & 0 & 3 & & 7 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 13 \end{bmatrix}.$$

Justify that you have provided all solutions.

In the next 4 problems, the matrices A and B are given by

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}.$$

Calculate the following whenever defined. Otherwise, state **undefined** and include a brief explanation.

(A9) (2 points) A^{-1}.

(A10) (2 points) AB.

(A11) (2 points) A matrix C such that $BC = I_2$ where I_2 is the 2×2 identity matrix.

(A12) (2 points) A matrix D such that $DB = I_3$ where I_3 is the 3×3 identity matrix..

(A13) (2 points) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\mathbf{x}) = M\mathbf{x}$ for some matrix M. Given that $M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $M \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the matrix A.

(A14) (2 points) Let $z = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i$. Express the complex number z^{2010} in the form $a + bi$, where a and b are real numbers.

(A15) (2 points) Assume that the matrix B was obtained by applying a sequence of elementary row operations to the matrix A. Circle **all** the statements that are true **independent of what the given A and B are.**

(a) $\det A = \det B$.

(b) If B is invertible then A is also invertible.

(c) If $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A then v is also an eigenvector of B.

(d) If $\lambda = 0$ is an eigenvalue of A then $\lambda = 0$ is also an eigenvalue of B.

(A16) (2 points) Let A, B be two $n \times n$ matrices, and let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^n$. Given that $A\mathbf{v} = B\mathbf{v}$, can we conclude that $A = B$? Justify your answer.

(A17) (2 points) Decide if transformation T – which we define below – is a linear transformation or not. Justify your answer.

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation that maps $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ into

$$T(\mathbf{x}) = \begin{bmatrix} -x_3 \\ 1 \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

(A18) (2 points) Find the matrix for the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \left(\begin{bmatrix} -y + z \\ x + y + z \\ -x - 2y \end{bmatrix}\right)$.

(A19) (2 points) The geometric problem of intersecting two planes P_1 and P_2 given respectively by the equations

$$x_1 + 2x_2 + 3x_3 + 4 = 0 \quad \text{and} \quad -2x_1 - 3x_2 - 4x_3 - 5 = 0$$

can be written as a matrix equation $A\mathbf{x} = \mathbf{b}$. Give a possible choice of A, $\mathbf{x}$ and $\mathbf{b}$.

(A20) (2 points) Let A, B, C and D be four size 3×3 invertible matrices. Circle all the options below that are equal to the matrix $(A(BC)^{-1}D)^T$ independent of what A, B, C and D are. No explanation required.

a) $A^T B^{-T} C^{-T} D^T$

b) $A^T (B^{-1})^T (C^{-1})^T D^T$

c) $A^T (C^{-1})^T (B^{-1})^T D^T$

d) $D^T (B^{-1})^T (C^{-1})^T A^T$

e) $D^T (C^{-1})^T (B^{-1})^T A^T$

(A21) (2 points) Calculate the determinant of the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 1 \\ 3 & 5 & 6 & 2 \\ -2 & 2 & 3 & 1 \end{bmatrix}$$

Consider a random walk with three possible states and where the random walker (possibly) changes from state to state once every day according to the transition matrix P. Answer the following two questions using the matrix P and some of its powers given below.

$$P = \begin{bmatrix} 0.0284 & 0.5938 & 0.1626 \\ 0.9207 & 0.1579 & 0.1634 \\ 0.0509 & 0.2483 & 0.6740 \end{bmatrix}, \qquad P^{10} = \begin{bmatrix} 0.2832 & 0.2907 & 0.2875 \\ 0.3843 & 0.3753 & 0.3786 \\ 0.3325 & 0.3340 & 0.3339 \end{bmatrix},$$

$$P^{20} = \begin{bmatrix} 0.2874 & 0.2875 & 0.2875 \\ 0.3791 & 0.3790 & 0.3790 \\ 0.3335 & 0.3335 & 0.3335 \end{bmatrix}$$

(A22) (2 points) What is the probability that a walker starting in location 3 is in location 1 after one time step?

(A23) (2 points) If you are the random walker and you want to maximize the probability of getting to location 2 when 10 days have passed from the beginning, what location should you start from?

(A24) (2 points) What is the probability that a walker that is in location 3 when 10 days have passed from the beginning is in location 1 when 20 days have passed from the beginning?

(A25) (2 points) What are the eigenvalues and eigenvectors of matrix A below?

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(A26) (2 points) Find the eigenvalues of matrix A where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(A27) (2 points) Construct a matrix with eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

(A28) (2 points) Suppose we run the following MATLAB code.

```
A=ones(4,4);
for j=1:4;
  A(j,:)=j*A(j,:);
end;
```

Specify the output if you now type:

```
>> A(2,4)
```

(A29) (2 points) Suppose we run the following MATLAB code.

```
a=1:2:5;
b=3:4:11;
A=[a;b];
```

Specify the matrix A that is produced by this code.

(A30) (2 points) Write a MATLAB for loop that generates a 20×20 matrix C given by

$$C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 20 \end{bmatrix}.$$

Part B: Long Answer Problems

(B1) Let L be the line $3x = 4y$.

(a) (2 points) Find a **unit** vector $\hat{\mathbf{a}}$ in the direction of L.

(b) (3 points) Find the matrix A which represents Ref_L, the reflection across the line L.

(c) (2 points) Calculate $\mathrm{Ref}_L\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$.

(d) (3 points) Find an eigenvector of A associated to the eigenvalue -1.

(B2) Let $A = (0, 1, 2)$, $B = (1, 0, 3)$, $C = (1, 1, 4)$. Let Δ be the triangle whose vertices are A, B, C.

(a) (2 points) Find the area of Δ.

(b) (2 points) Prove that Δ is a right-angled triangle.

(c) (3 points) Find the plane P containing Δ in the equation form, i.e., in the form

$$ax + by + cz = d$$

with a, b, c, d specified.

(d) (3 points) Find the point D on the plane P such that A, B, C, D form a rectangle.

(B3) Let A be a certain 2×3 matrix. Suppose that using elementary row operations the matrix A can be transformed into the matrix

$$U = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 4 \end{bmatrix}$$

(a) (4 points) Find all solutions, if any, of the system $A\mathbf{x} = 0$.

(b) (2 points) Can you find the solution of $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with the given information? Justify your answer.

(c) (4 points) Find all solutions, if any, of the system $A\mathbf{x} = A \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

(B4) Let A be the 3×3 matrix given by

$$A = \begin{bmatrix} -7 & 0 & 1 \\ 0 & 3 & 0 \\ -2 & 0 & -4 \end{bmatrix}$$

It is known that $\lambda_1 = 3$ is an eigenvalue of A.

(a) (2 points) Find an eigenvector associated with the eigenvalue $\lambda_1 = 3$.

(b) (4 points) Find the other eigenvalues λ_2 and λ_3 of A.

(c) (4 points) Find the eigenvectors associated with λ_2 and λ_3.

(B5) We are given the following transition matrix for a random walk:

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

(a) (3 points) What is the probability that a walker starting in location 1 is in location 1 after 2 steps?

(b) (5 points) Find $P^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) (2 points) Find $\lim_{n\to\infty} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(B6) Consider the sytem of linear differential equations given by

$$\begin{aligned} y_1'(t) &= 3y_1(t) - 2y_2(t) \\ y_2'(t) &= 4y_1(t) - y_2(t) \end{aligned}$$

(a) (2 points) Express this sytem in the form of $\mathbf{y}'(t) = A\mathbf{y}(t)$

(b) (6 points) Find the general solution of this system of differential equations *in real form.*

(c) (2 points) Find the the solution of the system of linear differential equations above with the initial condition $\mathbf{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.