Math 152 Final Exam (Spring 2012)

Last Name: ______________________________ **First name**: ______________________________

Student #: ______________________________ **Signature**: ______________________________

Circle your section #:

Gonzalez=202, M. Li=203, Friedman=205, Y. Li=206, Magyar=207, Pfeiffer=208

I have read and understood the instructions below:

Please sign:

Instructions:

1. The exam is closed-book exam. No notes or books or calculators are allowed.
2. Justify every answer whenever is necessary, and show your work. Unsupported answers will receive no credit.
3. All work you wish to be graded must be written on this exam booklet. Scrap work on additional papers/work books is allowed but will not be graded.
4. You will be given 2.5 hrs to write this exam. Read over the exam before you begin. You are asked to stay in your seat during the last 5 minutes of the exam, until all exams are collected.
5. At the end of the hour you will be given the instruction "Put away all writing implements and remain seated." *Continuing to write after this instruction will be considered as cheating.*
6. **Academic dishonesty:** Exposing your paper to another student, copying material from another student, or representing your work as that of another student constitutes academic dishonesty. Cases of academic dishonesty may lead to a zero grade in the exam, a zero grade in the course, and other measures, such as suspension from this university.

Question	grade	value
1		12
2		14
3		12
4		13
5		8
6		17
7		12
8		12
Total		**100**

Question 1: **[12 marks]**

Let S_1 be the plane perpendicular to $\mathbf{n} = (2, 1, -1)$ and through the point $P = (1, -1, 0)$. Let S_2 be the plane through the points $A = (1, 0, 0)$, $B = (-1, 2, 1)$, and $C = (0, 1, 1)$. By L we denote the line of intersection of S_1 and S_2.

(a) Determine the planes S_1 and S_2 in equation form.

(b) Determine L in equation form.

(c) Determine L in parametric form.

(d) Calculate the area of the triangle ABC.

(e) Calculate the volume of the parallelepiped with edges AB, AC, and AP.

(Question 1 continued here if more space is required.)

Question 2: **[14 marks]**

Consider the following system of linear equations

$$\begin{aligned} 2x - y + 2z &= a \\ x + 3y - 6z &= b \\ 3x + 2y - 4z &= c, \end{aligned}$$

where $a,\ b,\ c\ \in \mathbb{R}$,

(a) Write the system in matrix form $A\mathbf{x} = \mathbf{b}$. Clearly define A, $\mathbf{b}$, and $\mathbf{x}$.

(b) For what values of $a,\ b,\ c$ is the system homogeneous? Solve the resulting homogeneous system by calculating the reduced row echelon form of the augmented matrix.

(c) Given that $x = y = z = 1$ is a particular solution to the system given above, determine the values of $a,\ b$ and c. Then, find *all* the solutions in this case and express them in parametric form.

(Question 2 continued here if more space is required.)

Question 3: **[12 marks]**

Let A be the matrix for a linear transformation T. Suppose we know that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors of A associated to the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$, respectively.

(a) Express $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

(b) Calculate $T(\mathbf{e}_1)$.

(c) Find the matrix A.

Question 4: **[13 marks]**

Consider the matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

(a) Find the eigenvalues of A (there will be two distinct eigenvalues).

(b) For each eigenvalue of A, find an eigenvector.

(c) Use the information in items (a) and (b) to write A as MDM^{-1}, where M is an invertible matrix and D is a diagonal matrix.

(d) Use part (c) to explicitly compute A^k for any positive integer k.

(Question 4 continued here if more space is required.)

Question 5: **[8 marks]**

The following are two questions that do not require lengthy calculations. They are independent from each other.

(a) You wish to fit a line $y(x) = a + bx$ to the data points

$$(x_1, y_1) = (2, 2), \quad (x_2, y_2) = (3, 4), \quad (x_3, y_3) = (5, 6),$$

Write down the normal equations (check formula sheet if you do not remember) for the least squares approximation of the parameter vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ to the data. Then, write down the augmented matrix that corresponds to these equations one would use to compute the least squares approximation of a, b (but you don't need to compute a, b). Describe the least squares projection as a projection in three space onto a plane; specifically, which vector is being projected onto which plane?

(b) The following matrix A depends on a parameter p. Find all the values of the parameter p for which the corresponding matrix is not invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 100 & 200 & 300 & 401 \\ 0 & p & p+1 & p+2 \\ 0 & -p & -2p+1 & -3p+2 \end{bmatrix}.$$

Question 6: **[17 marks]**

These questions require little or no calculations. **Put your answers in the boxes.**

(a) For what value(s) of a are the following vectors $\mathbf{v}_1 = (1,\ 2,\ -1)$, $\mathbf{e}_2 = (0,\ 1,\ 3)$, $\mathbf{v}_3 = (a,\ 4,\ 5)$ linearly dependent?

(b) If $A = C^{-1}BC$, where C is an invertible matrix, and $\det B = 12$, $\det C = -3$. Calculate $\det A$.

(c) Let $A = M_{Rot_{60°}}$ be the matrix that rotates a 2D vector counterclockwise by 60°, $B = M_{Ref_{60°}}$ be the matrix that reflects in the line $y = \sqrt{3}x$, and $C = M_{Proj_{45°}}$ be the projection in the line $y = x$. Which three of the following six statements are true? (1) $A^6 = I$; (2) $A^{-1} = A$; (3) $B^3 = I$; (4) $B^{-1} = B$; (5) $C^{-1} = C$; (6) $C^2 = C$.

(d) Consider a linear system of 3 equations with 5 unknowns. Answer True (T) or False (F) to each of the following statements (no justification required). (1) It always has at least one solution. (2) There is either no solution or infinitely many solutions. (3) If a solution exists, then there is precisely a 2-parameter family of solutions. (4) If the coefficient matrix has rank k, then the associated homogeneous system has a k-parameter family of solutions.

	1	2	3	4
T/F				

(e) Is each of the following transformations linear? Answer 'Y' (yes) or 'N' (no) in the box (no justification required). (1) The reflection at the line $x = 0$ in the plane. (2) $T\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} v_1 + 2v_2 \\ -3v_1 \end{pmatrix}$. (3) $R\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} 2v_1 - 5v_2 \\ \cos^2(v_1) \end{pmatrix}$. (4) $S\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} -v_1 \\ v_2 v_3 \\ v_3 + v_1 \end{pmatrix}$.

	1	2	3	4
Y/N				

Question 7: **[12 marks]**

Consider a random walk that has transition matrix:

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 \\ 3/4 & 3/4 & 1/4 \\ 0 & 0 & 1/2 \end{pmatrix}$$

(a) If the walker starts in state 3 then find the probability that he will be in state 3 again after two steps.

(b) Find all eigenvalues of the transition matrix P (no need to calcculate eigenvectors at this stage).

(c) Find the equilibrium probability vector (i.e., the vector at which the probabilities remain unchanged from one step to the next).

(d) Based on results in (b), explain why the probability vector always approaches the equilibrium found in (c), regardless of the initial probability vector $\mathbf{x}(0)$.

(Question 7 continued here if more space is required.)

Question 8: **[12 marks]**

Consider the 2×2 system of differential equations

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t), \qquad \text{where } \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ is the vector for unknown functions.}$$

(a) Find all eigenvalues that may or may not be complex-valued.

(b) Find the corresponding eigenvectors.

(c) Based on results in (a) and (b), write down the general solution in real-valued form.

(d) Describe the trend of the solution as $t \to \infty$.

(Question 8 continued here if more space is required.)

Formula sheet:

1. For $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$ we have

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\vec{a}\| \, \|\vec{b}\| \, \cos(\theta), \quad \|\vec{a}\|^2 = \vec{a} \cdot \vec{a},$$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1), \quad \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin(\theta),$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1.$$

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1, \quad \mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}.$$

2. The least squares "best solution" to $B\vec{x} = \vec{c}$ is given by the "normal equations" $B^{\mathrm{T}} B \vec{x} = B^{\mathrm{T}} \vec{c}$.

3. The inverse of a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{is} \qquad M^{-1} = \frac{1}{\det M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

4. The eigenvalues of a 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are the solutions λ to the characteristic equation

$$\det[A - \lambda I] = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = 0 \qquad \text{or} \qquad \lambda^2 - \mathrm{tr} M \lambda + \det M = 0.$$

Eigenvectors $M\vec{x} = \lambda \vec{x}$ are solutions $\vec{x}$ to

$$(A - \lambda I)\vec{x} = \vec{0} \qquad \text{or} \qquad \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5.

$$M_{Rot_\theta} = \mathrm{RotCounterClock}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$M_{Ref_{\frac{\theta}{2}}} = \mathrm{ReflectHalfTheta}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$M_{Proj_\theta} = \mathrm{Proj2LineAtTheta2x}(\theta) = \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$$