The University of British Columbia

Final Examination - April 20, 2009

Mathematics 152

All Sections

Closed book examination. No calculators. Time: 2.5 hours

Last Name ________________ **First** ____________ **Signature** ________________

Section : ________________

Student Number ________________

Instructor : ________________

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.

  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

  (b) Speaking or communicating with other candidates.

  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

Part A - Short Answer Questions, 1 mark each

A1: Evaluate $(1, 2, -1) \times (4, -2, 1)$.

A2: A linear system of three equations in four unknowns has

- **(a)** always a unique solution.
- **(b)** either a unique solution or no solutions.
- **(c)** either a unique solution or an infinite number of solutions.
- **(d)** either no solutions or an infinite number of solutions.

A3: An electrical network with 2 voltage sources, 3 current sources, 5 resistors (all given) arranged in 6 elementary loops can be described using the loop current technique described in the notes and computer labs as a linear system with the following number of unknowns:

- **(a)** 6
- **(b)** 8
- **(c)** 9
- **(d)** 11

A4: An electrical network with 2 capacitors, 3 inductors, 5 resistors (all given) arranged in 6 elementary loops can be described as a system of the following number of differential equations:

- **(a)** 5
- **(b)** 6
- **(c)** 8
- **(d)** 16

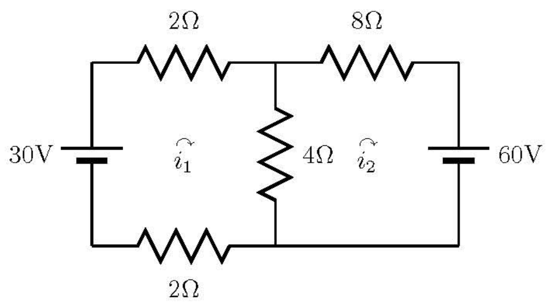
A5: Find the determinant of the matrix

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 2 \\ 0 & 1 & 8 \end{bmatrix}$$

A6: For what values of λ does the matrix

$$\begin{bmatrix} 3+\lambda & 2 \\ 2 & 3+\lambda \end{bmatrix}$$

not have an inverse?



A7: Consider the circuit above. Write a linear equation for the loop currents as shown in the figure that represent Kirchhoff's voltage law around the loop that corresponds to loop current i_1.

A8: Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A9: Are $\mathbf{a} = (1, 2, 4)$ and $\mathbf{b} = (6, -2, 3)$ orthogonal to each other? Justify briefly.

A10: What is the area of the parallelogram with sides given by the vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (2, 1)$?

A11: Do the vectors $(1, 1, 1)$, $(1, 2, 3)$, $(1, -7, 0)$ form a basis of $\mathbb{R}^3$? Justify briefly.

A12: You are solving a linear system of equations. You enter the system and right hand sides as an augmented matrix. The reduced row echelon form of this augmented matrix is given below:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Find all the solutions to the system (if any).

A13: What is the matrix representation of the 2D projection onto the x axis?

A14: The variables `x` and `y` are defined by the MATLAB commands

```
x = [1 0 2];
y = [3 2 1];
```

which of the following MATLAB commands will result in an error message?

(a) `dot(x,y)`

(b) `cross(x,y)`

(c) `x.*y`

(d) `x*y`

A15: The variable `A` is defined by the MATLAB command

```
A=[1 2 3; 4 5 6];
```

What is the result of the command `A(2,1)`?

A16: A matrix A is entered into MATLAB. The eigenanalysis of A is performed using the command `[T D] = eig(A)` which gives the following results:

```
T =
    0.7071    0.5257
    0.7071    0.8507
D =
    2.0000         0
         0    3.0000
```

Using these results, determine $A^5[1,1]^T$:

(a) $[32, 32]^T$

(b) $[32, 243]^T$

(c) $[1, 1]^T$

(d) $[243, 243]^T$

A17: The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T(x, y, z) = (2x + 2y, 3x + 3z, x + y + z).$$

Write the matrix representation of T.

A18: Write down the matrix A that will result from the following lines of MATLAB code:

```
A=zeros(3,3);
c=[1 2 3];
A(3,:) = -c;
for i=1:2
    A(i,i+1) = 1;
end
```

For questions **A19-A23** below, u and z are the complex numbers given below:

$$\begin{aligned} u &= i+1 \\ z &= 2-i \end{aligned}$$

A19: Evaluate $u+2z$. Your answer should be in the form $a+ib$ where a and b are real numbers.

A20: Evaluate $|z|$.

A21: What is the polar representation of u?

A22: Evaluate uz. Put your answer in the form $a+ib$ where a and b are real numbers.

A23: Evaluate u/z. Put your answer in the form $a+ib$ where a and b are real numbers.

A24: If $\mathbf{a} = (1,-1,1)$ and $\mathbf{b} = (2,3,4)$ find $\text{proj}_{\mathbf{a}}\mathbf{b}$.

A25: If $\mathbf{a} = (1, -1, 1)$ find the matrix representation of $\text{proj}_{\mathbf{a}}$.

A26. Find all roots z of

$$z^3 - 1 = 0$$

Put all your answers in the form $z = a + ib$ where a and b are real numbers.

A27: Consider the matrix which represents 2D reflection through the line $y = 10x$. What are the eigenvalues of this matrix?

A28: A is a 3×4 matrix. The entries of the first two rows are filled with random integers from 1 to 1000. The third row is the sum of the first two rows. What is the most likely value for the rank of A? Justify briefly.

A29: Write down the values of `x0` and `x1` that will result from the following lines of MATLAB code:

```
x0=1;
x1=1;
for i=1:3
   xnew = x0+x1;
   x0=x1;
   x1=xnew;
end
```

A30: $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation such that $\mathbf{x} = \mathbf{0}$ is the only vector $\mathbf{x}$ such that $T\mathbf{x} = \mathbf{0}$. What is the reduced row echelon form of the matrix representation of T?

Part B - Long Answer Questions, 5 marks each

B1: Three friends, Hiro, Wan and Bob together have \$16. Hiro has twice as much money as Bob and Wan has \$1 more than Hiro.

(a) [2 marks] Let $\mathbf{x} = (x_1, x_2, x_3)^T$ be the vector of unknowns, where x_1 is the amount of money that Hiro has, x_2 the amount that Wan has, x_3 the amount that Bob has. Describe the information above as a linear system in the form

$$A\mathbf{x} = \mathbf{b}$$

(write A and $\mathbf{b}$ with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. How much money do each of the three friends have?

B2: Let P be the plane defined by

$$x + 2y + 3z = 1$$

(a) [2 marks] Consider the intersection of P with a second plane Q defined by

$$2x - y - z = 0$$

This intersection is geometrically a line. Find a parametric description of this line.

(b) [3] Find the point on P closest to the point $(2, 0, 0)$.

B3: Consider

$$A = \begin{bmatrix} -2 & 0 & 3 \\ -3 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

(a) [2 marks] Find the eigenvalues of A. *Hint:* -4 is one of the eigenvalues.

(b) [3] Find a basis of eigenvectors of A.

B4: Consider the differential equation system

$$\begin{aligned} \frac{dy_1}{dt} &= ay_1 - 3y_2 \\ \frac{dy_2}{dt} &= -3y_1 + ay_2 \end{aligned}$$

where a is a real parameter.

(a) [1 point] Let

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

Write the matrix A so that the system above is written

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}.$$

(b) [2] Find the eigenvalues and eigenvectors of A when $a = 1$.

(c) [1] Write the general solution to the differential equation system when $a = 1$.

(d) [1] For what values of a (if any) do *all* solutions of the system satisfy $y_1(t) \to 0$ and $y_2(t) \to 0$ as $t \to \infty$?

B5: Consider the differential equation system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where A has eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$ with corresponding eigenvectors

$$\mathbf{k}_1 = [1 + i, 1 - i]^T, \text{ and } \mathbf{k}_2 = [1 - i, 1 + i]^T$$

(a) [2 marks] Write the general solution of the DE system.

(b) [3] Find the solution (written in terms of real functions of t) that satisfies initial conditions $\mathbf{x}(0) = [1, 2]^T$.

B6: The matrix

$$P = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

is the transition matrix for a random walk.

(a) [1 mark]. If the random walk starts in the second state, what is the probability that it will be in the second state after 2 transitions?

(b) [2] Find the equilibrium probability of the random walk represented by P.

(c) [2] Find a different 2×2 matrix Q that is a transition matrix for a random walk such that

$$\lim_{n \to \infty} Q^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

does not exist.

The End