The University of British Columbia

Final Examination - December 17, 2008

Mathematics 102

Instructors (Sections) **(Circle one)**:

Drs. L. Keshet (101 / 104), K. Gunn (102), R. Israel (103) I. Rozada (105), J. Allard (106)

Closed book examination Time: 2.5 hours

First Name: ____________________ **Family name:** ______________________________

Student #: ______________________ **Signature** _________________________________

Special Instructions: - Be sure that this examination has 13 pages. **The last page contains useful formulae** Write your *full* name (as on your Student ID) on top of each page. You may use backs of pages for scrap work, (or extra space, if needed).

- No calculators, electronic devices, books, or notes are permitted.
- Unless otherwise indicated, show all your work. Answers not supported by calculations or reasoning may not receive credit. Messy work will not be graded.
- At the end of the examination period: 1. You will be instructed to put away all writing implements (Continuing to write past this signal is considered cheating). 2. Remain in your seats until exams have been collected. 3. You will be instructed when you are free to leave.

1		15
2		12
3		10
4		10
5		10
6		12
7		11
8		10
9		10
Total		100

Problem 1: Multiple Choice Questions: Select ONE correct answer (a, b, c, d, or e) and write it in Table 1. You will not be graded for any work or answers outside those boxes.

1.1: The equation of the tangent line to the function $y = f(x)$ at the point x_0 is

(a) $y = f(x_0) + f'(x_0)(x - x_0)$
(b) $y = f'(x_0) + f(x_0)(x - x_0)$
(c) $y = x_0 + f(x_0)/f'(x_0)$
(d) $y = f(x) - f'(x)(x - x_0)$
(e) $y = f(x_0) - f'(x_0)(x - x_0)$

1.2: The functions $f(x) = x^2$ and $g(x) = x^3$ are equal at $x = 0$ and at $x = 1$. Between $x = 0$ and $x = 1$, for what value of x are their graphs furthest apart?

(a) $x = 1/2$ (b) $x = 1/3$ (c) $x = 2/3$ (d) $x = 1/4$ (e) $x = 3/4$

1.3: Consider a point in the first quadrant on the hyperbola $x^2 - y^2 = 1$ with $x = 2$. The slope of the tangent line at that point is

(a) $2/\sqrt{5}$ (b) $1/\sqrt{3}$ (c) $\sqrt{5}/2$ (d) $2/\sqrt{3}$ (e) $2/3$

1.4: For $a, b > 0$, solving the equation $\ln(x) = 2\ln(a) - 3\ln(b)$ for x leads to

(a) $x = e^{2a-3b}$ (b) $x = a^2/b^3$ (c) $x = 2a - 3b$ (d) $x = a^2b^3$ (e) $x = (a/b)^6$

1.5: The function $y = f(x) = \arctan(x) - (x/2)$ has local maxima (LX), local minima (LM) and inflection points (IP) as follows:

(a) LX: $x = -1$, LM: $x = 1$, IP: $x = 0$. (b) LX: $x = \sqrt{3}$, LM: $x = -\sqrt{3}$, IP: $x = 0$.
(c) LX: $x = 1$, LM: $x = -1$, IP: none (d) LX: $x = 1$, LM: $x = -1$, IP: $x = 0$.
(e) LX: $x = -\sqrt{3}$, LM: $x = \sqrt{3}$, IP: $x = 0$.

Answers to Multiple choice questions should go in the boxes below. Use this page for scrap.

Q 1.1	Q 1.2	Q 1.3	Q 1.4	Q 1.5

Table 1: Fill in these boxes with the letters (a, b, c, d, or e) corresponding to the one correct answer for each question. **Illegible or ambiguous responses will not receive marks.** NOTE: carefully check to ensure that you have correctly matched the response with the relevant questions. Only answers in this table using letters a, b, c, d, or e will be graded for Problem 1.

Problem 2: The following problems require little or no computation. Put the answer in the box provided. You may use space here for supporting reasoning or computations, if any, for which part-marks would be awarded.

(a) Suppose $y = f(x)$ is a function having the following properties: $f(1) = 2, f'(1) = 3$. Suppose $g(x)$ is the *inverse* function for $f(x)$. Then the slope of the tangent line to $g(x)$ at the point $x = 2$ is:

slope= [ ]

(b) A differential equation describing radioactive decay of a substance is

$$\frac{dy}{dt} = -ky,$$

where $y(t) = Ce^{-kt}$ is the amount of radioactivity remaining. A plot of $y(t)$ versus t is found to be a curve that goes through the point $(0, 10)$, and has tangent line of slope -5 at that point. What are the values of the constants?

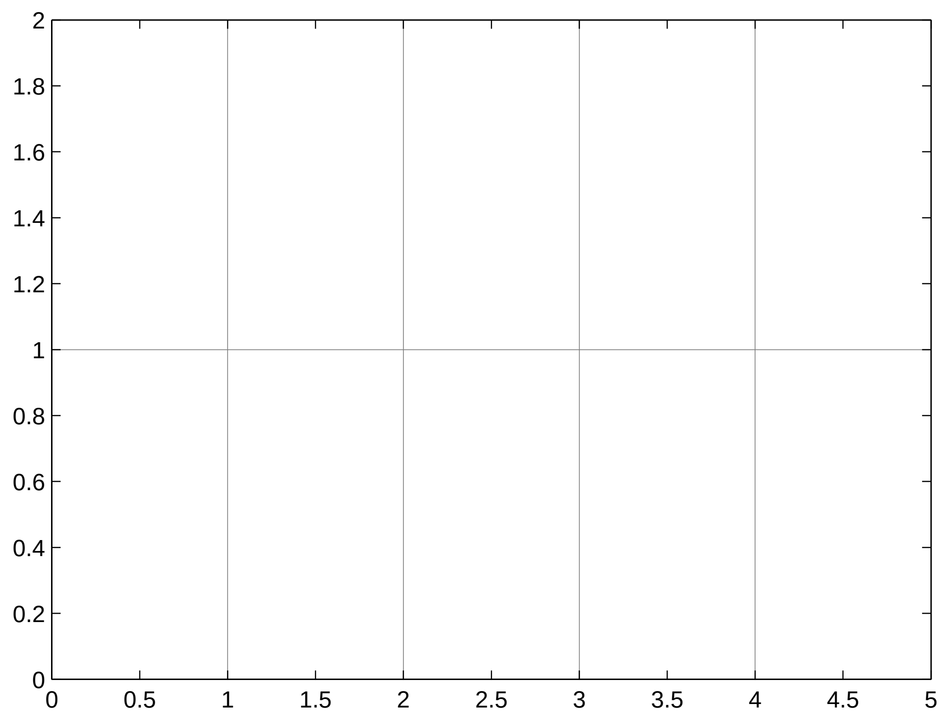
$C =$ [ ], $k =$ [ ]

(c) A linear approximation for $y = \sin(0.05)$ that uses the fact that $\sin(0) = 0$ is

$\sin(0.05) \approx$ [ ]

Problem 3: The graph of a certain smooth function $f(x)$ has the following properties: $f(0) = f(5) = 1$; there are critical points at $x = 1$ and $x = 3$ and points of inflection at $x = 2$ and $x = 4$; there are no other critical points or inflection points, and the graph is concave up for $x \leq 2$. Inside the interval $0 < x < 5$, the maximum value of $f(x)$ is 2 and the minimum value of $f(x)$ is 0. Sketch the graph on the grid provided.

[**Suggestion: use a pencil for your first attempt, for easier modifications.**]



Problem 4:

Two spherical balloons are connected so that one inflates as the other deflates, the sum of their volumes remaining constant. When the first balloon has radius 10 cm and its radius is increasing at 3 cm/sec, the second balloon has radius 20 cm. What is the rate of change of the radius of the second balloon at that time?

[Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$].

Rate of change of radius of second balloon = $\boxed{}$

Problem 5:
A particle's motion is described by $y(t) = t^3 - 6t^2 + 9t$
where $y(t)$ is the *displacement* (in metres), t is time (in seconds), and $0 \leq t \leq 4$ seconds.

[**3pts**] (a) During this time interval, when is the particle farthest from its initial position?

At time(s) =

[**4pts**] (b) During the time interval $0 \leq t \leq 4$, what is the greatest speed (in either direction) of the particle?

Greatest speed =

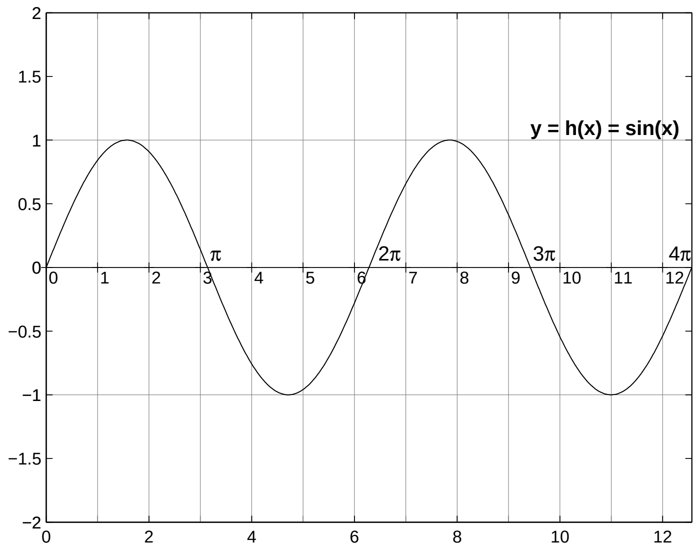
[**3pts**] (c) What is the total *distance* (including both forward and backward directions) that the particle has travelled during this time interval?

Total distance =

Problem 6: Consider the two functions

$$h(x) = \sin(x), \quad g(x) = ax, \quad x \geq 0,$$

where $a > 0$ is a constant. $y = h(x)$ is shown in the graph below. Your answers to parts (a) and (b) should include typical sketches of $y = g(x)$.
[**Note the relationship of this question to Lab 6 in your Math 102 course**.]



Suppose $a = 2$. In that case, the only intersection is at $x = 0$.

(a) For what range of values of a will there be more than one intersection for $x \geq 0$ [i.e. one at $x = 0$ and also intersection(s) at some other point(s)]?

$a <$ ☐

Sketch an example of $y = ax$ for such a on the same graph.

(b) Name any positive value of a such that there are exactly four intersection points for $x \geq 0$ (including the one at $x = 0$).

Sketch an example of $y = ax$ for such a on the same graph. $a =$ ☐

(c) Now suppose that $a = 1/2$. How many intersection points are there for $x > 0$?
Use Newton's method with the initial guess $x_0 = \pi$ to find an improved estimate, x_1 for the point of intersection of the two graphs. [Leave your answer as a simplified fraction in terms of π. Do not compute a decimal approximation.]

Number of intersections = ☐

$x_1 =$ ☐

Problem 7: "Live and Learn"
Knowledge can be acquired by studying, but it is forgotten over time. A simple model for learning represents the amount of knowledge, $y(t)$, that a person has at time t (in years) by a differential equation

$$\frac{dy}{dt} = S - fy$$

where $S \geq 0$ is the rate of studying and $f \geq 0$ is the rate of forgetting. We will assume that S and f are constants that could have different values for different people. [Your answers to the following questions will contain constants such as S or f.]

[3pts](a) Mary never forgets anything. What does this imply about the constants S and/or f?

Mary starts studying in school at time $t = 0$ with no knowledge at all. How much knowledge will she have after 4 years (i.e. at $t = 4$)?

$y(4) =$ ☐

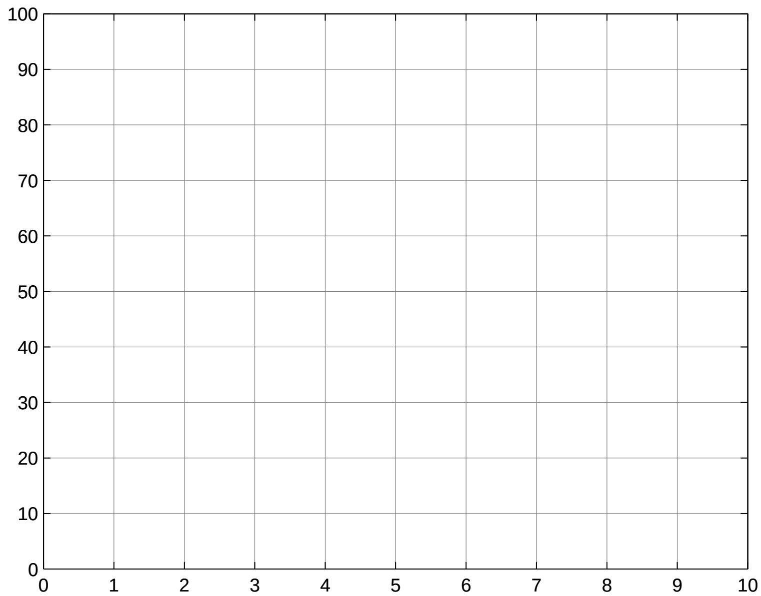
[3pts](b) Tom learned so much in preschool that his knowledge when entering school at time $t = 0$ is $y = 100$. However, once Tom in school, he stops studying completely. What does this imply about the constants S and/or f?

How long will it take him to forget 75% of what he knew?

Time = ☐

Problem 7, Continued

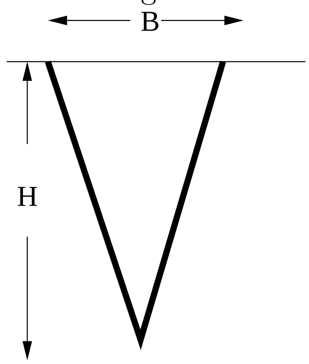
[5pts](c) Jane studies at a rate of 10 units per year and forgets at a rate of 0.2 units per year. Sketch a "direction field" ("slope field") for the differential equation describing Jane's knowledge. Add a few curves $y(t)$ to show how Jane's knowledge changes with time.



How much knowledge would Jane have if she keeps studying (and forgetting) for a very long time?

Jane's knowledge after a long time = ☐

Problem 8: Egyptian plovers are small birds that eat food off the exposed surface of crocodile teeth. The crocodile benefits from clean teeth, while the plover gets a free meal. Assume that the exposed crocodile teeth are isosceles triangles (two equal sides) and that the exposed perimeter (total length of dark lines in figure) is always $P = 10$ cm. The height of the teeth varies (short teeth at the back of the mouth, tall teeth at the front). The plover wants to choose the tooth with largest surface area (and thus, the largest amount of free food). What height of tooth should the plover look for?



Crocodile tooth

The exposed perimeter, P, of a tooth of base B and height H is

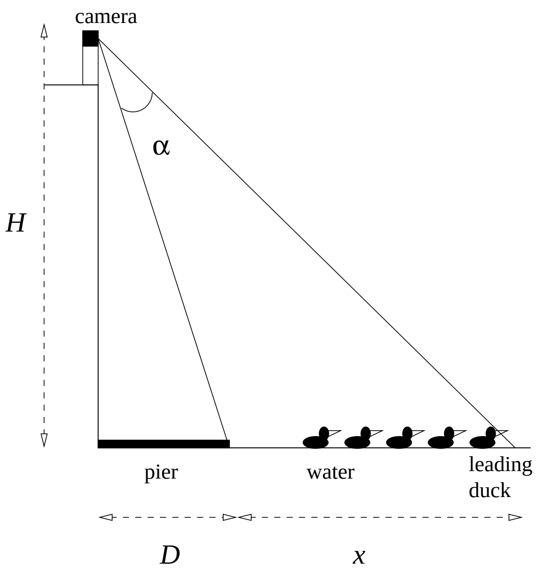
$$P = 2\sqrt{H^2 + (B/2)^2}.$$

The exposed surface area, A, of the tooth is

$$A = \frac{1}{2}BH.$$

Height of tooth = $\boxed{}$

Problem 9:



Graduate student Ryan Lukeman studies behaviour of duck flocks swimming near Canada Place in Vancouver, BC. This figure from his PhD thesis shows his photography set-up. Here $H = 10$ m is the height from sea level up to his camera at the observation point, $D = 2$ m is the width of the pier (a stationary platform whose size is fixed), and x is the distance from the pier edge to the leading duck in the flock (in metres). α is a visual angle (at the camera) between the line of sight to the leading duck and to the pier edge, as shown in the figure.

At the instant that the visual angle is increasing at a rate of 1/100 radians per second, at what rate is the leading duck swimming away from the pier if it is 3m away from the pier?

speed of leading duck = [ ]

Useful Formulae

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Trig identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Values:

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
π	0	-1