Math 101—Final Examination

April 21, 2017

Duration: 150 minutes

____________________ Surname (Last Name) | ____________________ Given Name | ____________________ Signature

____________________ Student Number | ____________________ Section

Do not open this test until instructed to do so! This exam should have 12 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam. Phones cannot be visible at any point during the exam.

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	8		7	7	
2	8		8	7	
3	6		9	7	
4	6		10	7	
5	6		11	7	
6	6		**Total**	75	

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

1a. **[2 pts]** Evaluate $\int_{-1}^{2} (1 - |x|)\ dx$. Answer: [ ]

A: $5/2$
B: $3/2$
C: $-1/2$
D: $1/2$
E: 0
H: 1

1b. **[2 pts]** Evaluate $\int \frac{3x}{\sqrt{x^2 - 3}}\ dx$. Answer: [ ]

J: $3\sqrt{x^2 - 3} + C$
K: $\frac{3x}{\sqrt{x^2 - 3}} + C$
L: $\frac{3}{2}\sqrt{x} + C$
M: $\frac{3}{\sqrt{x^2 - 3}} + C$
O: $\frac{3}{2}\sqrt{x^2 - 3} + C$
P: $3\sqrt{x} + C$

1c. **[2 pts]** For which real numbers x does the power series $\sum_{n=0}^{\infty} \frac{(2x-3)^n}{n+1}$ converge?

Answer: [ ]

Q: $1 < x \leq 2$
R: all real numbers x
S: $-\frac{3}{2} < x \leq \frac{3}{2}$
T: only $x = \frac{3}{2}$
U: $1 \leq x < 2$
W: $-\frac{3}{2} \leq x < \frac{3}{2}$

1d. **[2 pts]** Let y be the solution to the differential equation $\frac{dy}{dx} = \frac{\cos x}{2y}$ satisfying $y(0) = \sqrt{3}$.

What is $y(\frac{\pi}{2})$? Simplify your answer completely. Answer: [ ]

Problems 1–2 are answer-only questions: the correct answers in the given boxes earns full credit, and no partial credit is given.

2a. **[2 pts]** Which integral represents the area to the right of the curve $y^2 = 1 - x$ and to the left of the curve $2y^2 = 5 - x$? Answer: ☐

A: $2\int_0^2 (4 - y^2)\,dy$
B: $\int_1^5 \left[\sqrt{(5-x)/2} - \sqrt{1-x}\right] dx$
C: $\int_1^5 \left[\sqrt{1-x} - \sqrt{(5-x)/2}\right] dx$
D: $\int_1^5 \left[(1-x) - (5-x)/2\right] dx$
E: $\int_{-2}^2 (y^2 - 4)\,dy$

2b. **[2 pts]** Which of the following substitutions is most helpful in evaluating the integral $\displaystyle\int_2^4 \frac{dx}{x^2\sqrt{x^2+2x+10}}$? Answer: ☐

F: $x = \sqrt{10}\tan u$
G: $u = x^2 + 2x + 10$
H: $u = x^2$
J: $x = 3\sec u - 1$
K: $x = 3\tan u - 1$
L: $x = \sqrt{10}\sec u$

2c. **[4 pts]** For each of the following series, choose the appropriate statement. (Write **N**, **O**, **P**, **S**, or **T** in each box; each answer will be used at most once, and each series matches a single answer only.)

N: The series converges by the Ratio Test.
O: The series converges absolutely by the Comparison Test with a p-series.
P: The series converges by the Alternating Series Test.
S: The series diverges.
T: The series converges by the Integral Test.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ Answer: ☐

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{7^{n+3}}{3^{2n} - 2}$ Answer: ☐

(iii) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \arctan n$ Answer: ☐

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^3}$ Answer: ☐

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

3a. **[3 pts]** Find the area of the region between the curves $y = x - 8$ and $y = -x^2 + 7x - 16$. A calculator-ready answer is sufficient.

3b. **[3 pts]** The interval of convergence of the power series $\sum_{n=0}^{\infty} nx^{n+1}$ is $(-1, 1)$. (You don't have to show this.) Find a formula (not involving infinite series) for $\sum_{n=0}^{\infty} nx^{n+1}$, valid for all $-1 < x < 1$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

4a. **[3 pts]** Let R be the region above the curve $y = 3x^4$ and below the line $y = 3$. The area of the region R is $\frac{24}{5}$ (you do not have to prove that). Find the centroid of the region R. Simplify your answer completely.

4b. **[3 pts]** Let M_6 denote the Midpoint Rule approximation with $n = 6$ points. Find an upper bound for the difference between the integral $\displaystyle\int_0^{\pi} x \sin x \, dx$ and its approximation M_6. You may use the fact that, when approximating $\displaystyle\int_a^b f(x)\,dx$ with the Midpoint Rule using n points, the absolute value of the error is at most $\dfrac{K(b-a)^3}{24n^2}$, where $|f''(x)| \leq K$ for all $a \leq x \leq b$.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

5a. **[3 pts]** Evaluate $\displaystyle\int_0^{\pi} \sin^5 x \, dx$. A calculator-ready answer is sufficient.

5b. **[3 pts]** Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2}$. (Hint: interpret the sum as a Riemann sum.) Simplify your answer fully.

Problems 3–6 are short-answer questions: put a box around your final answer, but you must include the correct accompanying work to receive full credit.

6a. **[3 pts]** Find the slope of the tangent line to the curve $y = \int_{1-x}^{1+2x} e^{t^2}\, dt$ at the point $(0, 0)$. Simplify your answer completely.

6b. **[3 pts]** For which values of x does the series $\sum_{n=1}^{\infty} \frac{2}{3^n (x-1)^{n+1}}$ converge?

Problems 7–11 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

7. Let R be the region to the right of the y-axis and to the left of the curve $x^2 = \sqrt{4+y}$, with y between -4 and 0. A reservoir dug into the ground has the shape that results from the region R being rotated around the y-axis.

(a) **[3 pts]** Calculate the volume of the reservoir (by slicing horizontally). Simplify your answer completely.

(b) **[4 pts]** The reservoir from part (a) is filled with a fluid of density $1234\ \mathrm{kg/m^3}$. Find the work, in joules, required to pump the fluid out of the top of the reservoir. Use $g = 9.8\ \mathrm{m/s^2}$ for the acceleration due to gravity. A calculator-ready answer is sufficient.

8. **[7 pts]** Evaluate $\int_{-3}^{-4} \frac{x-8}{x^3-4x^2+4x}\,dx$. A calculator-ready answer is sufficient.

9. **[7 pts]** Evaluate $\int_0^\infty 6x^3 e^{-x^2}\, dx$. Simplify your answer completely.

10. **[7 pts]** Evaluate $\lim_{x\to 0} \frac{\cos x - e^{-x^2/2}}{x^3 \sin x}$. A calculator-ready answer is sufficient.

11. Let $\{A_n\}$ be a sequence of positive numbers such that the power series $\sum_{n=0}^{\infty} A_n(x-1)^n$ has radius of convergence $R = \frac{3}{2}$.

(a) **[3 pts]** Does the series $\sum_{n=0}^{\infty} A_n$ converge or diverge? Justify your answer.

(b) **[2 pts]** Does the sequence $\left\{\frac{1}{A_n}\right\}$ converge or diverge? Justify your answer.

(c) **[2 pts]** Does the series $\sum_{n=0}^{\infty} n(n-1)A_n$ converge or diverge? Justify your answer.