Spectral Geometry of Liouville Quantum Gravity

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Spectral geometry of LQG ?

- LQG is a certain canonical random geometry arising naturally in physics. What do classical theorems of (spectral) geometry become in this context?

- What can be said about eigenvalues, eigenfunctions in LQG?

- Eg: **Can you hear the shape of Liouville quantum gravity?**

- Connections to ``**quantum chaos".**

Mark Kac

Plan

- I: Background on Liouville theory:
  Gaussian free field, Gaussian multiplicative chaos.

- II: Liouville Brownian motion: a canonical diffusion in LQG
  Definition, spectrum, etc.

- III: Reminders on spectral geometry
  Classical *Weyl law*

- IV: Results on spectral geometry of LQG,

- V: Conjectures
  Quantum chaos

I. Background on Liouville theory

- **Polyakov** (1981):

  Quantisation of the Liouville Lagrangian, motivated by **2D quantum gravity**

  Given (Σ, g) a 2D Riemannian manifold,

  $$\text{for } \phi : \Sigma \to \mathbb{R}, \quad S(\phi) = \frac{1}{4\pi}\int_\Sigma [\,|\nabla\phi(z)|^2 + QR_g(z)\phi(z) + \mu e^{\gamma\phi(z)}]\mathrm{d}v_g(z)$$

  Scalar curvature

  Riemannian volume

  $Q = \gamma/2 + 2/\gamma$

  Cosmological constant

  $\gamma \in (0,2)$ = coupling constant

- **Liouville** Conformal Field Theory:

  $$\mathbf{P}(\mathrm{d}\phi) = \exp(-S(\phi))\mathrm{D}\phi$$

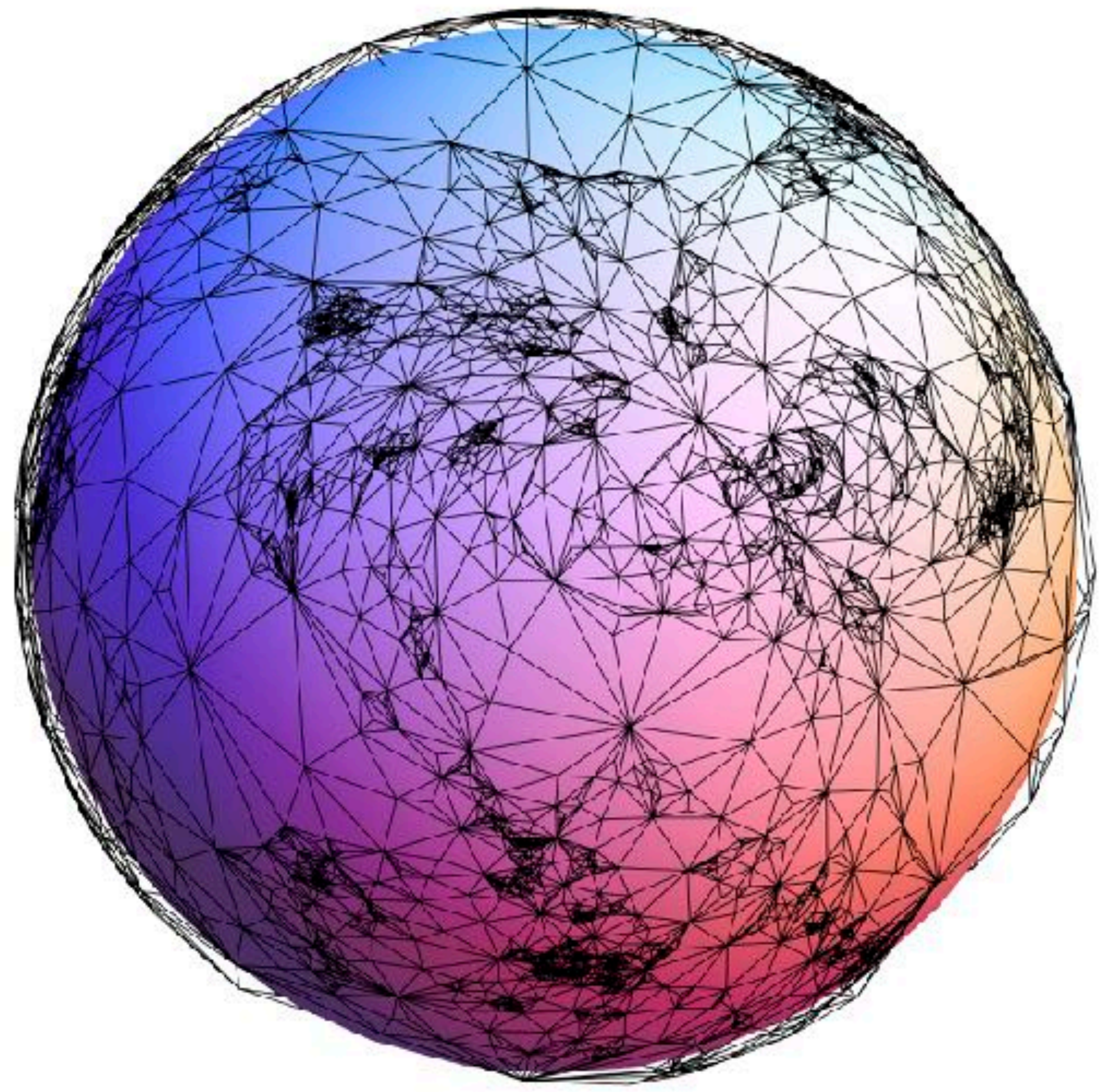
  "uniform" measure on fields

I. Background

- **Rigorous definition** initiated by Duplantier, Sheffield (2010) then fully by David, Kupiainen, Rhodes, Vargas (2016)
- Toy model: ϕ = **Gaussian free field** in surface Σ or domain D, with Dirichlet boundary conditions.
- Think of ϕ as random **conformal factor.** *Very* informally,

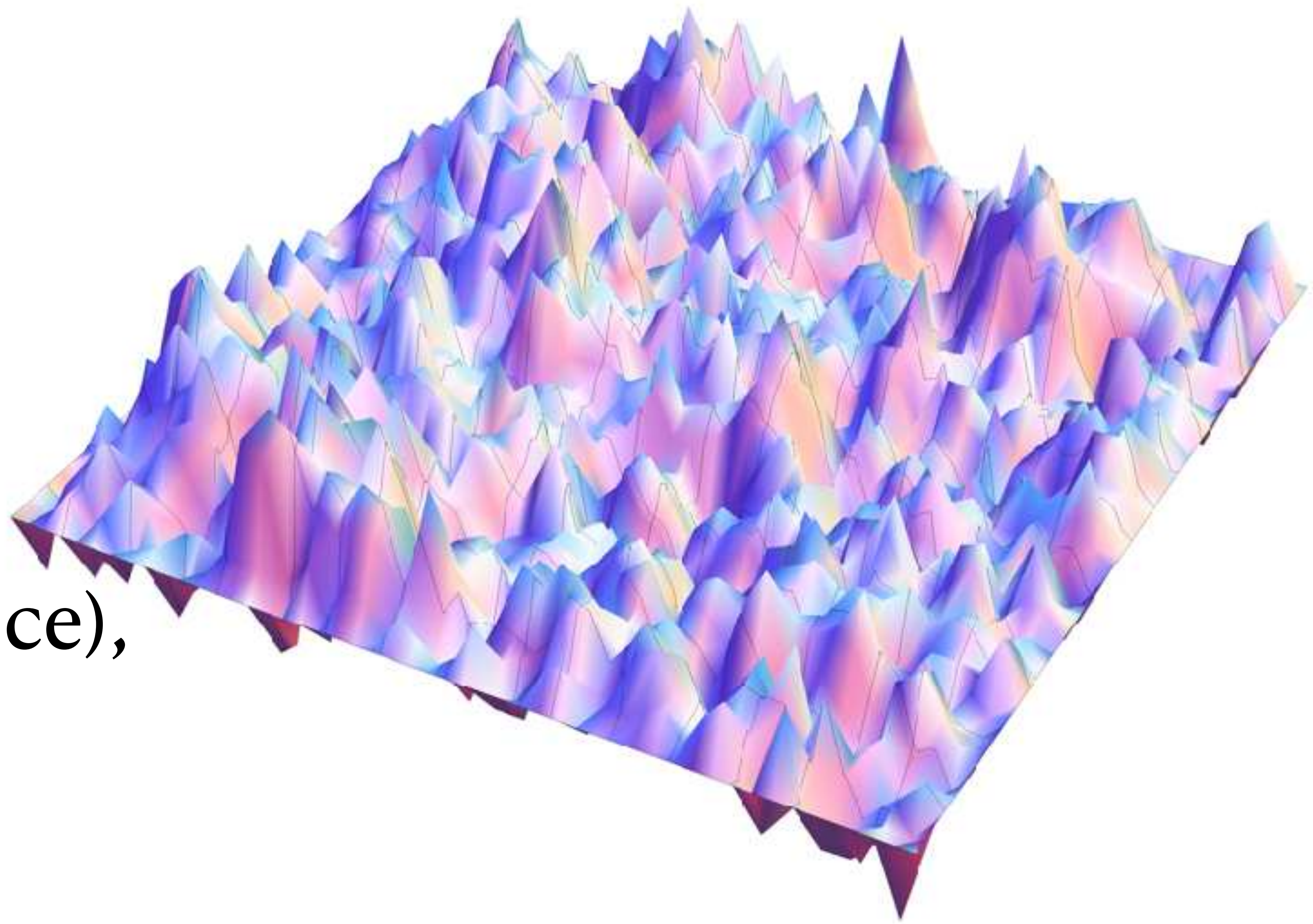
$$\mathrm{dist}(a,b) = \inf_{\eta} \int_0^1 e^{\gamma\phi(\eta(t))} |\eta'(t)| \, \mathrm{d}t$$

$$\mathrm{vol}(A) = \int_A e^{\gamma\phi(x)} \mathrm{d}x: \qquad \gamma \in (0,2) = \textbf{coupling constant.}$$



Continuum GFF

- $D \subset \mathbb{R}^2$: bounded domain. Let
  $$G_D(x,y) = \pi \int_0^\infty p_t^D(x,y)\mathrm{d}t$$
  be the (continuum) **Green function with Dirichlet boundary conditions**.
- In two dimensions, (owing to neighbourhood recurrence),
  $G_D(x,y) = -\log|x-y| + O(1)$ as $|y-x| \to 0$
  (**logarithmic blowup)**.
  Nice away from diagonal.
- As a result, we **cannot** define $(\phi(x))_{x\in D}$ as a Gaussian centered stochastic process $\mathbb{E}[\phi(x)\phi(y)] = G_D(x,y)$



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Continuum GFF

- Instead, we view ϕ as a **Gaussian stochastic process** $\big((\phi, f)\big)_{f \in \mathscr{D}(D)}$, indexed by test functions $f \in \mathscr{D}(D)$ (=smooth with compact support):
$$\mathbb{E}[(\phi, f)(\phi, g)] = \iint_D f(x) G_D(x, y) g(y) \ \mathrm{d}x \ \mathrm{d}y$$
(Definition of this process via Kolmogorov's extension theorem.)
- In fact, this extends to f which can be **rougher**: $f = f^+ - f^-$, with $f^\pm$ nonnegative measures such that
$$\iint G_D(x, y) f^\pm(\mathrm{d}x) f^\pm(\mathrm{d}y) < \infty.$$
(say $f \in \mathscr{M}$ for such allowed test signed measures). Analytically: $f \in H^{-1}(D)$.

Continuum GFF

- Reminder: $f \in \mathcal{M}$ if $\displaystyle\iint G_D(x, y) f^{\pm}(\mathrm{d}x) f^{\pm}(\mathrm{d}y) < \infty$.
- Ex: $f \in \mathcal{D}(D)$. Does $f \in \mathcal{M}$?
- **Yes!**
- Ex: $f =$ uniform measure on circle contained in D?
- **Yes!**
- Ex: $f = \delta_{x_0}(\,\cdot\,)$?
- **No!**

Circle Averages

- Let $z \in D$, $\varepsilon > 0$ such that $B(z, \varepsilon) \subset D$. Let $\phi_\varepsilon(z)$ = circle average of ϕ at distance ε from z.
- This is well defined, and is a nice **regularisation** of GFF at scale ε.
- Then $\phi_\varepsilon(z)$ is Gaussian with variance $\log(1/\varepsilon) + O(1)$ (**logarithmic blowup**).
- In fact, if $B_t = \phi_{e^{-t}}(z)$, then $(B_t, t \geq t_0)$ is a **1D Brownian motion !**

Thick Points

- Thick points are exceptional points of the GFF. They play an important role.
- Def: a point $z \in D$ is called $\alpha-$**thick** if

$$\lim_{\varepsilon \to 0} \frac{\phi_\varepsilon(z)}{\log(1/\varepsilon)} = \alpha.$$

- Call $\mathcal{T}_\alpha$ the set of thick points. Note that for fixed $\alpha > 0$, and given $z_0 \in D$, $z_0 \notin \mathcal{T}_\alpha$, almost surely.
- Indeed, $\lim_{t \to \infty} \frac{B_t}{t} = 0$, almost surely.

Thick Points

- Nevertheless, $\mathscr{T}_\alpha$ is not necessarily empty: there can be **exceptional points** for which the BM has drift α !
- In fact, $\dim_H(\mathscr{T}_\alpha) = (2 - \alpha^2/2)_+$, and $\mathscr{T}_\alpha \neq \varnothing$ **if and only if** $\alpha \leq 2$.
- Explanation/heuristics:

$$\mathbb{E}[\#\{z : \phi_\varepsilon(z) \approx \alpha \log(1/\varepsilon)\}] = \varepsilon^{-2}\mathbb{P}[\phi_\varepsilon(z) \approx \alpha \log(1/\varepsilon)]$$

$$\lesssim \varepsilon^{-2} \exp\left(-\frac{(\alpha \log(1/\varepsilon))^2}{2 \log(1/\varepsilon)}\right)$$

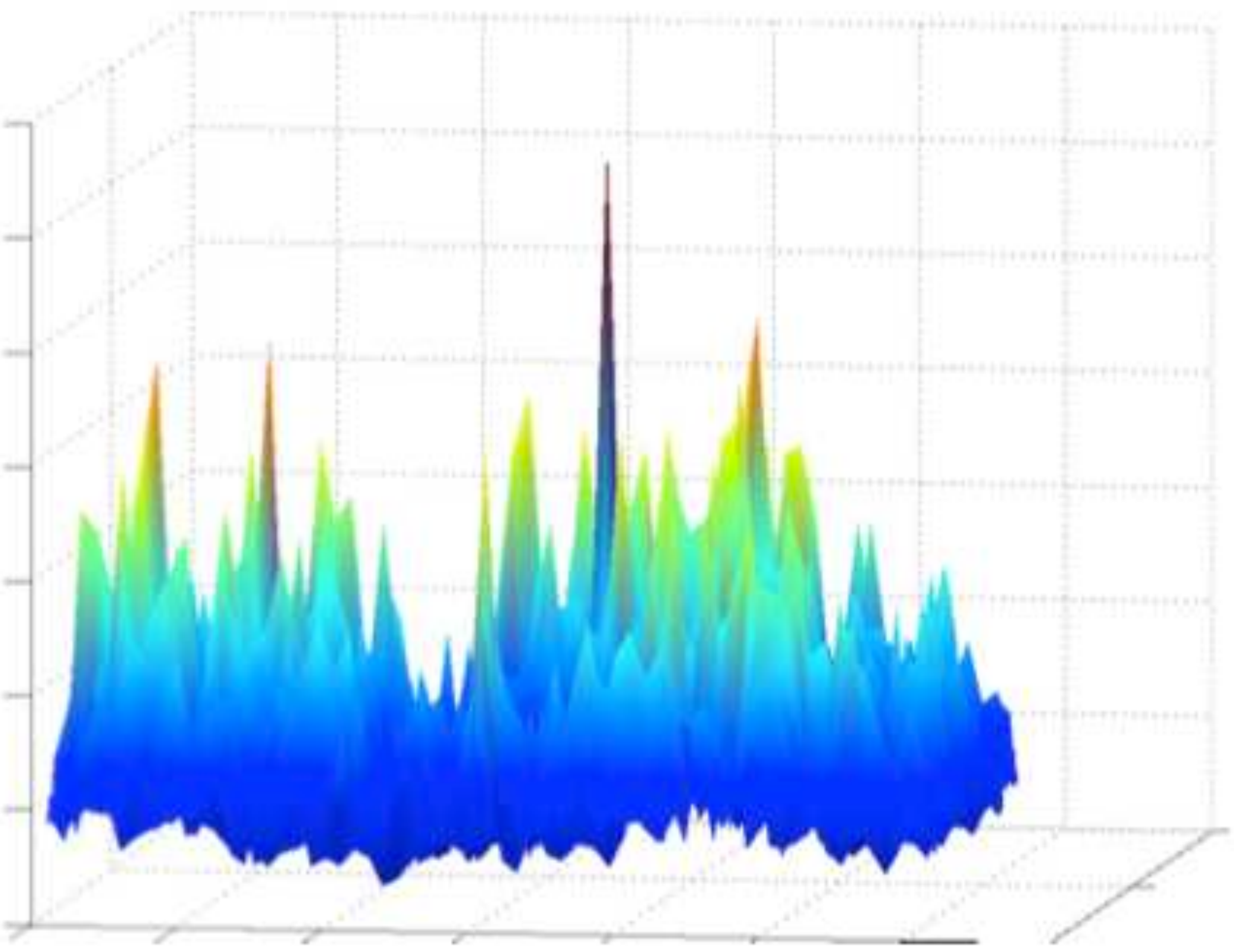
$$= \varepsilon^{-2+\alpha^2/2}$$

on an ε-mesh grid

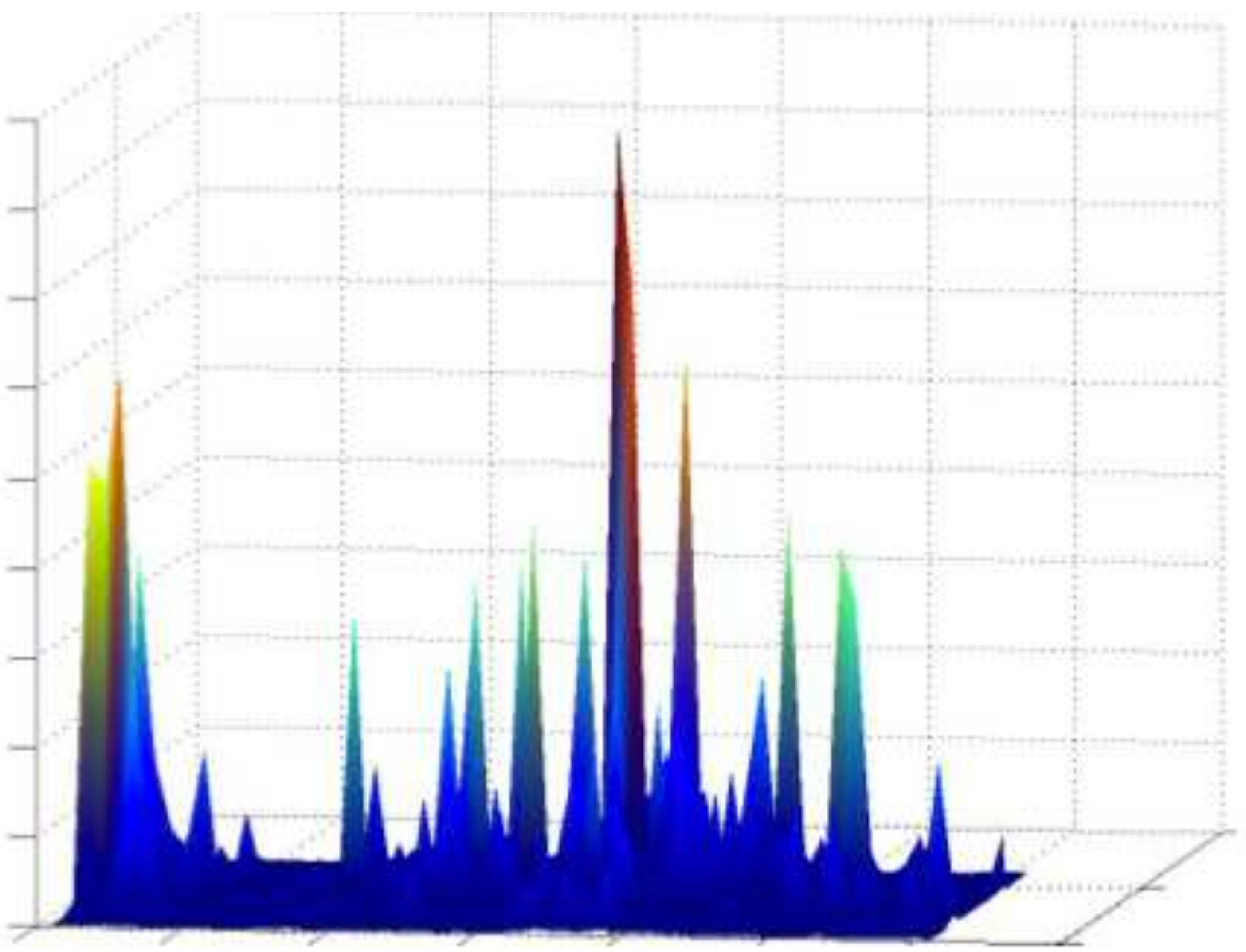
Gaussian Multiplicative Chaos

- Introduced by **Kahane** 80s, motivated by **turbulence** (Kolmogorov, Mandelbrot).
- Let ϕ denote a GFF on domain $D \subset \mathbb{R}^2$ (more generally: **log-correlated** Gaussian field in $\mathbb{R}^d$). $\phi_\epsilon =$ some regularisation of ϕ at scale ϵ: $\phi_\epsilon = \phi * \theta_\epsilon$ for some convolution kernel θ with compact support.
- Let $M_\epsilon(\mathrm{d}x) = \epsilon^{\gamma^2/2} e^{\gamma \phi_\epsilon(x)} \mathrm{d}x$, where $\gamma \geq 0$ is a **coupling constant**.
- **Theorem** (Kahane '85, B. '17, Shamov '17)
  For $0 < \gamma < 2$, $\lim_{\epsilon \to 0} M_\epsilon(\mathrm{d}x)$ exists in probability w.r.t. weak topology.
  Limit M = **GMC** is nonzero iff $\gamma < 2$ (more generally: $\gamma < \sqrt{2d}$).
  Universal: does not depend on θ.

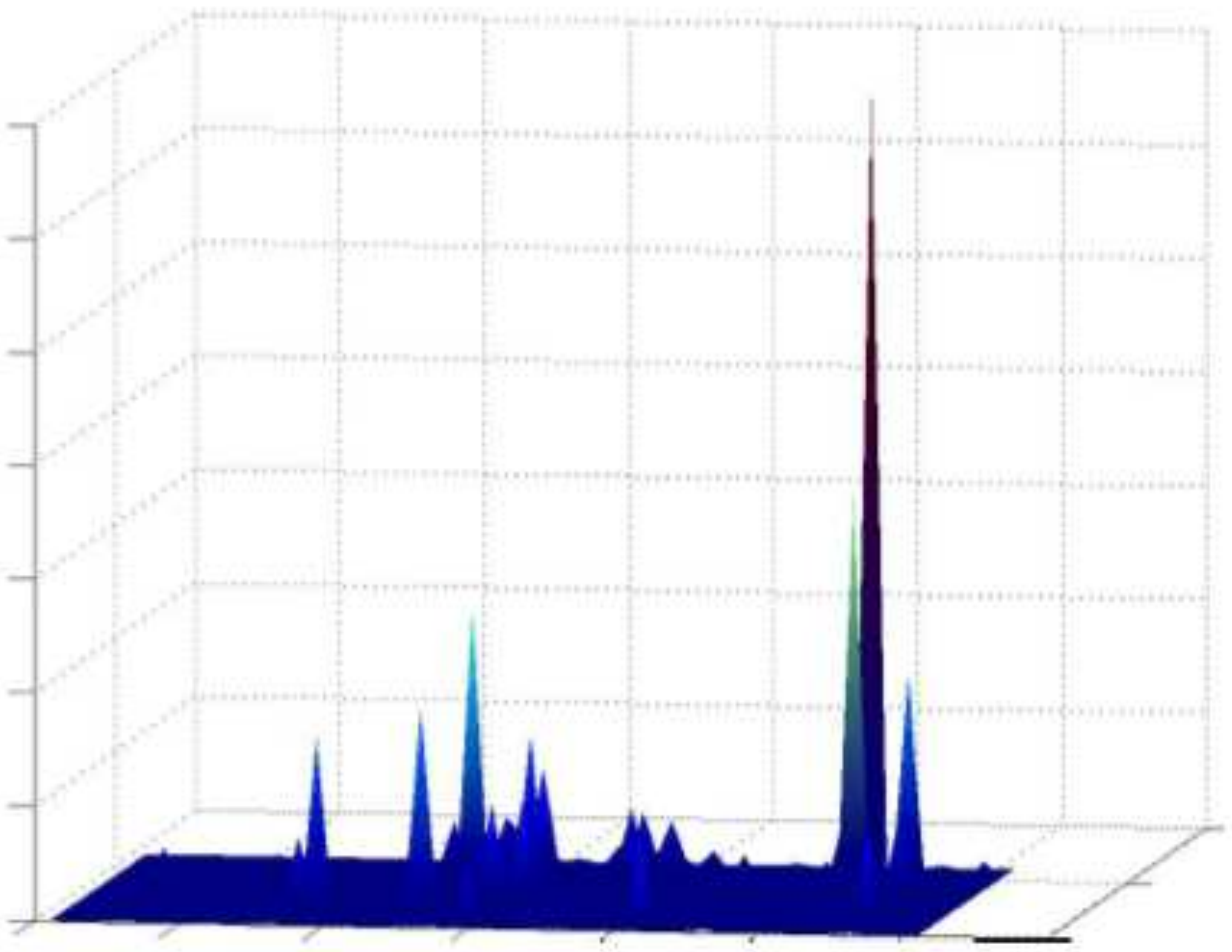
Gaussian Multiplicative Chaos



$\gamma = 0.2$



$\gamma = 1$



$\gamma = 1.8$

Gaussian Multiplicative Chaos

- One can show: M is a.s. supported on $\gamma-$thick points.
- That is, $M(\mathcal{T}_\gamma^c) = 0$, a.s.
- Sampling from M, points are a.s. $\gamma-$thick.
- This is why the measure cannot exist when $\gamma > 2$. (Recall $\mathcal{T}_\alpha = \emptyset$ if $\alpha > 2$).

Liouville Quantum Gravity

- Recall: (Σ, g) a 2D Riemannian manifold,

$$\text{for } \phi : \Sigma \to \mathbb{R}, \quad S(\phi) = \frac{1}{4\pi} \int_\Sigma [\,|\nabla \phi(z)|^2 + QR_g(z)\phi(z) + \mu e^{\gamma\phi(z)}]\mathrm{d}v_g(z)$$

- **Liouville** Conformal Field Theory:

$$\mathbf{P}(\mathrm{d}\phi) = \exp(-S(\phi))\mathrm{D}\phi\,?$$

- **Idea** (going back constructive field theory Glimm—Jaffe 1970s):

$$\exp\Big(-\int_\Sigma |\nabla \phi|^2 \mathrm{d}v_g\Big)D\phi := \mathbb{P}^{\mathrm{GFF}}(\mathrm{d}\phi)$$

Liouville Quantum Gravity

- Then (oversimplification of work of David-Kupiainen-Rhodes-Vargas 2016)

$$P(\mathrm{d}\phi) = \exp\left(-\int_\Sigma QR_g(z)\phi(z)v_g(\mathrm{d}z) - \mu M(\Sigma) \right) \mathbb{P}^{GFF}(\mathrm{d}\phi)$$

  gives a well defined measure ! (Where $M(\Sigma) =$ total mass of GMC measure.)

- If Σ is the sphere, there are no boundary conditions and this introduces additional complications.

- **Correlation functions** can be computed exactly: **DOZZ formula** (Kupiainen, Rhodes, Vargas, Ann. Math.).
- There are close connections to random planar maps (cf. **Nina Holden's course**).

Toy model

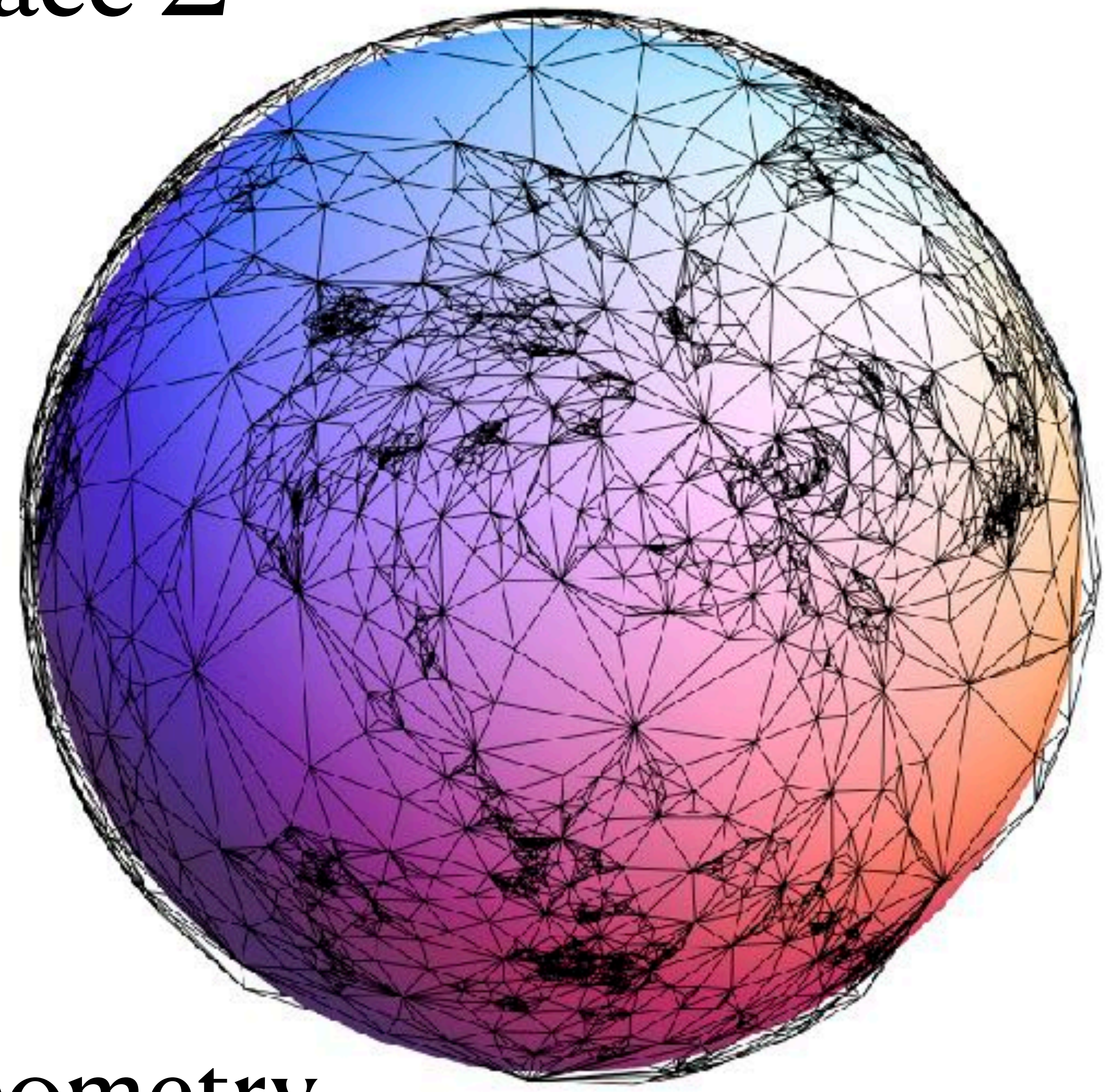
- In the rest of these lectures, ϕ will simply have the law $\mathbb{P}^{GFF}(\mathrm{d}\phi)$ (instead of Polyakov's normalized measure) on a domain $D \subset \mathbb{R}^2$ instead of surface Σ

- Recall: ϕ endows domain D with random geometry: informally

$$\mathrm{dist}(a,b) = \inf_{\eta} \int_0^1 e^{\gamma\phi(\eta(t))}|\eta'(t)|\,\mathrm{d}t$$ (NB: hard to define rigorously)

$$\mathrm{vol}(A) = \int_A e^{\gamma h(x)}\mathrm{d}x = M(A):$$

GMC plays the role of the uniform volume measure in this random geometry.



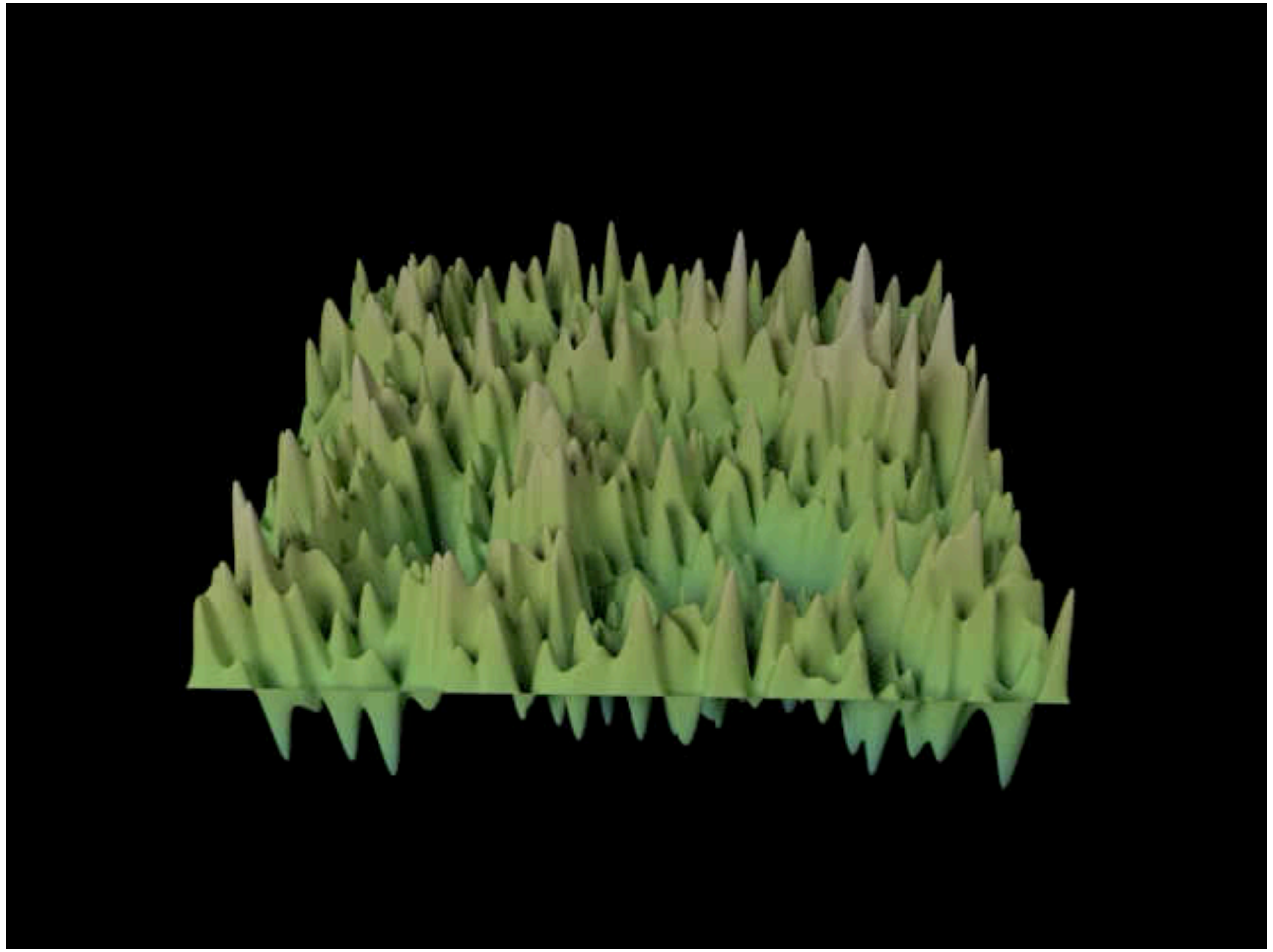
II - Liouville Brownian motion

- Spectrum of **Laplace-Beltrami** operator, but diffusion more natural to describe.

- **Theorem** (B. '15); (Garban-Rhodes-Vargas '15). Existence of Liouville Brownian Motion:

  $Z(t) = \lim_{\varepsilon \to 0} Z_\varepsilon(t)$, where

  $$Z_\varepsilon(t) = B_{F_\varepsilon^{-1}(t)}; F_\varepsilon(t) = \varepsilon^{\gamma^2/2} \int_0^t e^{\gamma \phi_\varepsilon(B_s)} ds,$$



© H. Jackson. Landscape = ϕ, where ϕ is a GFF.
Riemannian volume "=" $M(dx) = e^{\gamma \phi(x)} v_g(dx)$.

Liouville Brownian motion

- Equivalent definitions:
  Z is a time-change of Brownian motion, where the **PCAF** has **Revuz measure** $= M$
- **Dirichlet form**: $\mathcal{E}(f, g) = \int_D \nabla f(x) \cdot \nabla g(x) \mathrm{d}x$, with respect to $L^2(M)$.
- Think of scaling limit of SRW on triangulation.
- **Theorem** (B.-Gwynne '20): SRW on certain planar maps converge to Liouville Brownian motion.

Liouville Brownian motion

- LBM $(Z(t))_{t\leq\tau_D}$ is a.s. **continuous**, does not stay **stuck** (iff $0 \leq \gamma < 2$).
- Can be started a.s. from *all* points **simultaneously** (given ϕ).
- Forms a.s. a **Feller process**. (Garban-Rhodes-Vargas).
- Leaves GMC measure M **invariant** (e.g. on the sphere, torus).
- For each $t \geq 0$, a.s. $Z(t) \in \mathcal{T}_\gamma$.
- A.s., $t \mapsto Z(t)$ is a.e. **differentiable** if $\gamma > \sqrt{2}$, with $Z'(t) = 0$ (Jackson '17) !
- Scaling limit of random walk on random planar maps (CRT-mated maps), B.-Gwynne '2020
- $\mathbf{P}_t(x, \cdot\,) \ll M$. So the Radon-Nikodym derivative exists and is called the **heat kernel**. (Garban-Rhodes-Vargas).

III. Spectral Geometry

- We now recall a few facts from spectral geometry, before discussing what we know in LQG.

- **Schuster** (1882): ``it would baffle the most skillful mathematician to find out the shape of a bell by means of the sound which it is capable of sending out''.

Spectral Geometry

- **Lorentz** (1910) in Göttingen:

  "In an encolosure with a perfectly reflecting surface there can form standing electromagnetic waves, analogous to tones of an organ pipe. We shall confine our attention to very high overtones. [...] There arises the mathematical problem to prove that the number of overtones which lie between frequencies ν and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume... It has been verified for many simple shapes... There is no doubt that the theorem holds in general.."

- **Hilbert** (apocryphal): not to be solved in my lifetime !

Weyl's law (1912)

- **Theorem**. Let $N(\lambda) = \sum_{n\geq 1} 1_{\{\lambda_n \leq \lambda\}} =$ eigenvalue counting function. Then, as $\lambda \to \infty$,

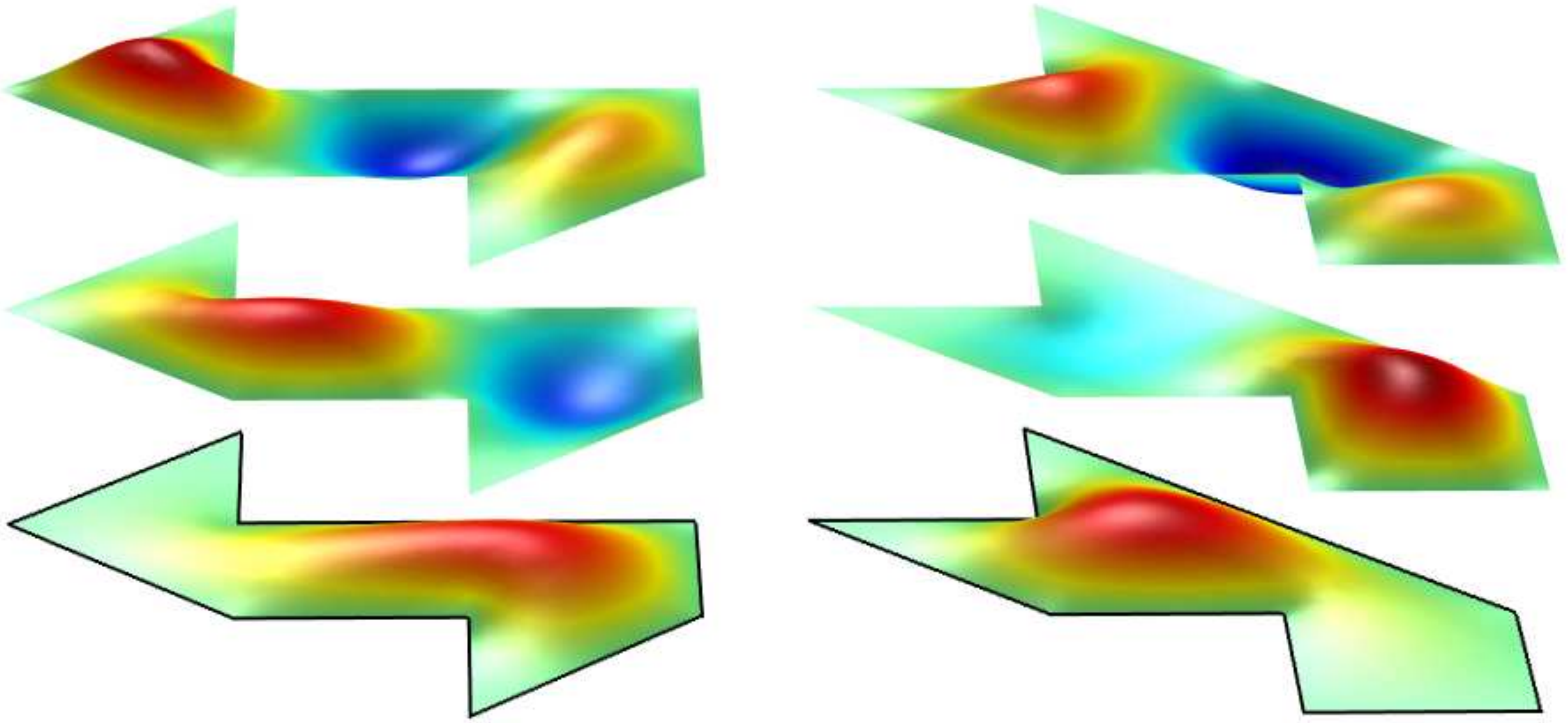
$$\frac{N(\lambda)}{\lambda} \to c_0 \mathrm{Leb}(D), \quad c_0 = \frac{1}{2\pi}.$$

- This result would be the same whatever 2D Riemannanian manifold.
(Of course, also works with adjustments when in dimension $D \geq 2$.)
- **Q:** What do these results become in LQG?

Can you hear the shape of a drum?

- **Kac 1966**: do eigenvalues $\{\lambda_n\}_{n\geq 0}$ of $-\frac{1}{2}\Delta$ determine uniquely the domain (up to isometetry)?

- Known counterexamples in Riemannian world (Milnor; **Gordon-Webb-Wolpert** 92)



Wikipedia

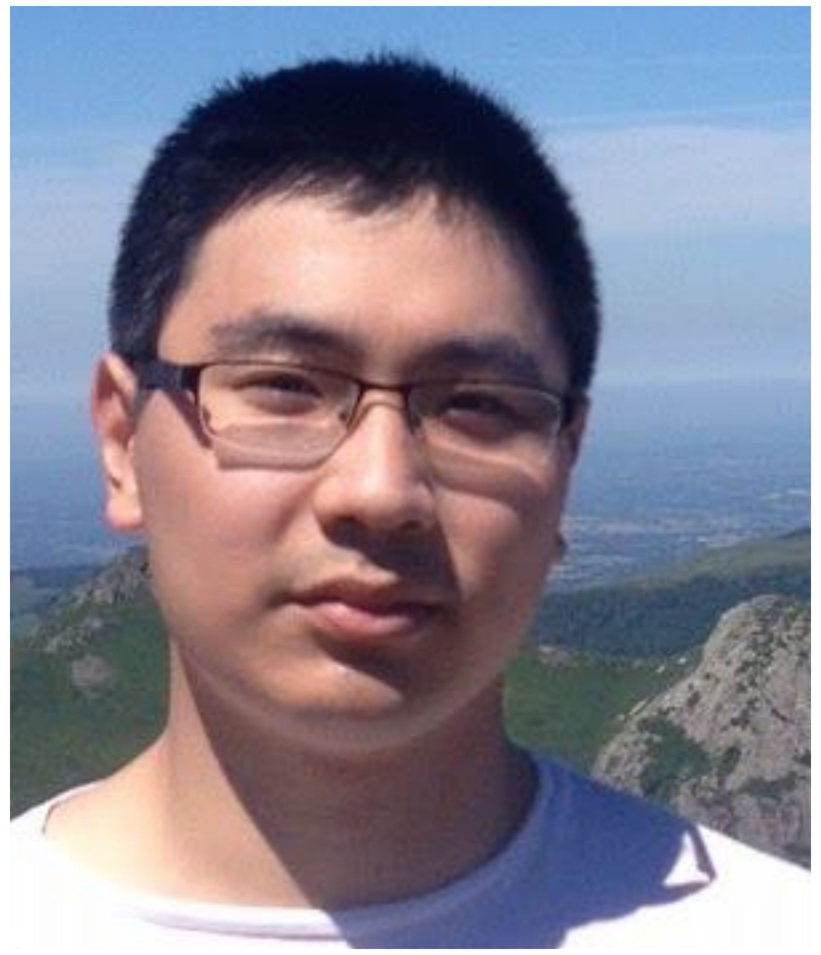
IV. Main results

- Back to LQG. We will study its **spectrum**. How is this defined?
- Answer:

Andres-Kajino '16,
Maillard-Rhodes-Vargas-Zeitouni '16

- The **infinitesimal generator** of LBM is delicate to handle directly
- But the Green function $\mathbf{G}(x, dy)$ is a.s. a nice compact operator on $L^2(M)$
  $\rightarrow$ apply the spectral theorem to it.
- Get a.s. $\{\mathbf{f}_n\}_{n\geq 1}$, ON basis of eigenfunctions for $L^2(M)$, with EV = $\boldsymbol{\lambda}_n$ (random), $\mathbf{G}\mathbf{f}_n(x) = \frac{1}{\lambda_n}\mathbf{f}_n(x)$.
- Let $\mathbf{N}(\lambda) = \sum_n 1_{\{\lambda_n \leq \lambda\}}$, **eigenvalue counting function**

LQG Weyl law



Mo Dick Wong

- **Theorem** (B.-Wong 2023):

  Let ϕ = Dirichlet GFF in domain $D \subset \mathbb{R}^2$. Let $\gamma \in (0,2)$.

  As $\lambda \to \infty$, $\dfrac{\mathbf{N}(\lambda)}{\lambda} \to c_\gamma M(D)$,

- where $c_\gamma = \dfrac{1}{\pi}\left\{ \mathbb{E}\left[\int_0^\infty \mathcal{I}\left(e^{\gamma(B_t - \alpha t)}\right) dt\right] + \mathbb{E}\left[\int_0^\infty \mathcal{I}\left(e^{-\gamma \mathcal{B}_t^\alpha}\right) dt\right]\right\}$, $\mathcal{I}(x) = xe^{-x}$,

  $\alpha = Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2} > 0$, $\mathcal{B}^\alpha$ = BM with drift α conditioned to stay positive.

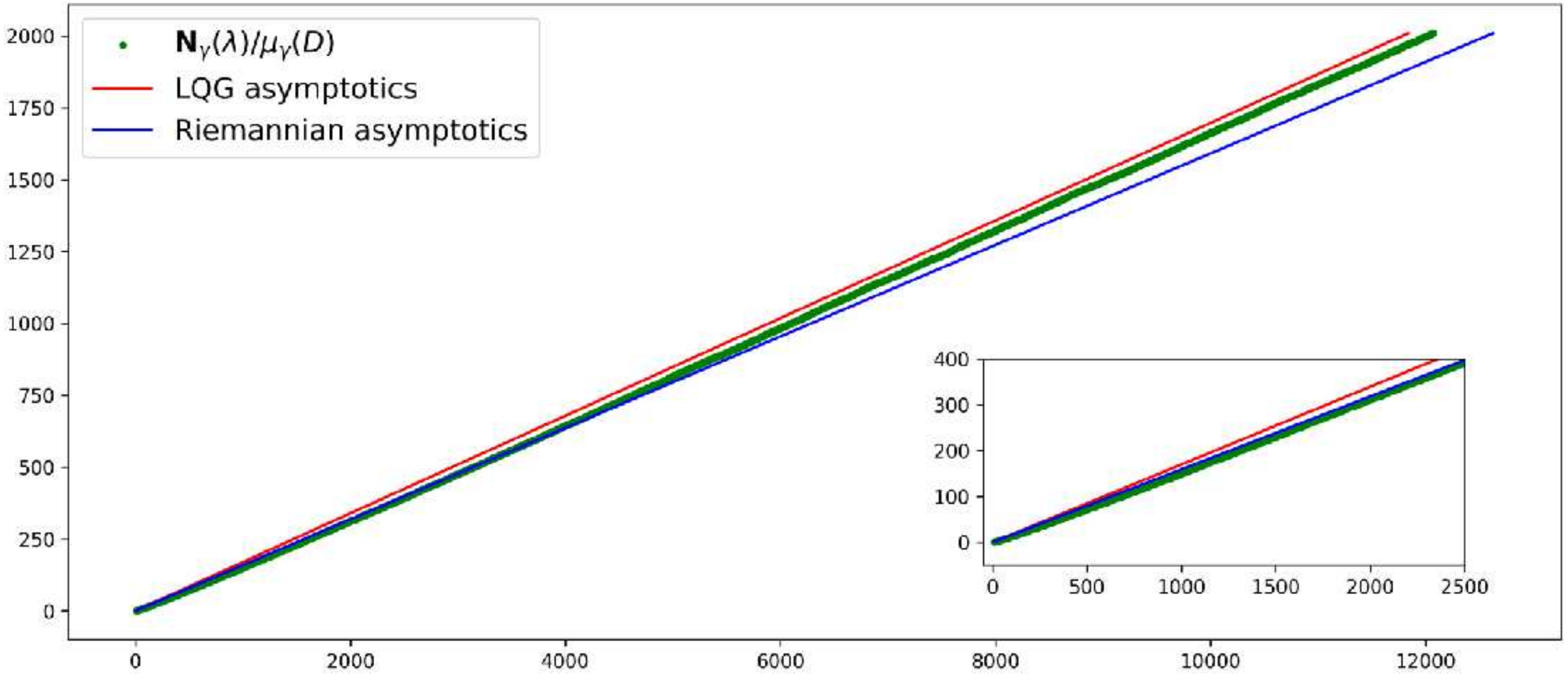
- Concisely, $c_\gamma = \dfrac{1}{\pi}\mathbb{E}[\int_{-\infty}^\infty \mathcal{I}(e^{\gamma C(t)})dt]$ where $C(t) =$ **Sheffield's quantum cone.**

Weyl law constant

- **Theorem** (**B.-Wong** 2023):

$$c_\gamma = \frac{1}{\pi(2-\gamma^2/2)}.$$

Note that $c_\gamma > c_0$, for all $\gamma \in (0,2)$!



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Trace Formula

- Starting point: spectral decomposition: $\mathbf{p}_t(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\mathbf{f}_n(x)\mathbf{f}_n(y)$,

  where $\mathbf{p}_t(x,y) =$ Liouville **heat kernel**, which is a.s. jointly continuous (not obvious!).

- Take $x = y$, and integrate over $x \in D$:

- $$\mathbf{H}(t) := \int_D \mathbf{p}_t(x,x)M(dx) = \sum_{n=1}^{\infty} e^{-\lambda_n t} = \int_0^{\infty} e^{-\lambda t} d\mathbf{N}(\lambda),$$

  where $\mathbf{N}(\lambda) = \sum_{n=1}^{\infty} 1_{\{\lambda_n \le \lambda\}}$ is the eigenvalue counting function of LQG.

- Thus **heat trace** = Laplace transform of $\mathbf{N}(\lambda)$.

Heat kernel asymptotics

- To prove the result we use the **trace formula** and show that for any open set $A \subset D$,
$$t \int_A \mathbf{p}_t(x,x) M(\mathrm{d}x) \to c_\gamma M(A), \text{ as } t \to 0,$$
in probability.

- Usually, very hard to work with $\mathbf{p}_t(x,y)$.
BUT: here **bridge decomposition**
$$\int_0^\infty g(t)\ \mathbf{p}_t(x,y)\mathrm{d}t = \int_0^\infty \mathbf{E}_{x\to y;t}\ [g(F(t))]\ p_t(x,y)\mathrm{d}t$$
where $F(t)$ is the quantum clock = time-change.

Heat kernel asymptotics

- We established: $t\int_A \mathbf{p}_t(x,x)M(\mathrm{d}x) \to c_\gamma M(A)$ as $t \to 0$ for *all* open sets $A \subset D$.

- **Q:** is it the case that $\mathbf{p}_t(x,x) \sim \dfrac{c_\gamma}{t}$ as $t \to 0$, $M-$a.e.?

- *Cons*: multifractal geometry.
  Pros: restrict to typical points.

Heat kernel asymptotics

- **Ans**: we can prove that it is **not** the case, even for $\mu-$a.e. $x \in D$.
- Sample x from $M(\mathrm{d}x)$, view $\mathbf{p}_t(x,x)$ as a RV. Average over randomness of GFF (=*annealed asymptotics*).
- **Theorem**: (B.-Wong '23 conjectured, B.-Klein '25+)
  $t\mathbf{p}_t(x,x) \to X$ in law (=annealed), a nontrivial random variable as $t \to 0$.
  In fact, $\big(t\mathbf{p}_t(x,x), t\mathbf{p}_t(y,y)\big) \to (X, Y)$ independent.
- We expect **logarithmic** upper and lower **pointwise fluctuations,** even if we restrict to Liouville typical points.

Second Term in Weyl's law

Joint work with Jakob Klein

Jakob Klein

- Back in 1912, Weyl **conjectured**:
  $N(\lambda) = c_0 \lambda \mathrm{Leb}(D) - c_0' \sqrt{\lambda}\, |\partial D| + o(\sqrt{\lambda})$.

- This is still open! (**Ivrii 1981**: up to an ergodic assumption, but hard to verify in practice)

- ``Corresponding'' **heat trace** expansion is known:

$$H(t) = \int_D p_t(x,x)dx = c_0 \frac{|D|}{t} - c_0' \frac{|\partial D|}{\sqrt{t}} + \dots$$

(in particular, $|\partial D|$ is **spectrally determined**)

Anomalous heat trace expansion

- Setup: $\phi =$ Sheffield's $\gamma-$**quantum cone**, restricted to **bounded smooth** $D \subset \mathbb{R}^2$.
  Informally: $\phi(z) = \mathrm{GFF}_{\mathbb{C}}(z) + \gamma \log(1/|z|)$.
- Describes scaling limit of whole plane models of maps (e.g., UIPT) when $\gamma = \sqrt{8/3}$.
  Alternatively, local limit of LQG when we ``zoom in''. (Metric geometry: **tangent cone)**.
- **Theorem** (B.-Klein, '25+)
  Let $\mathbf{H}(t) = \int_D \mathbf{p}_t^D(x,x)\ M(\mathrm{d}x) =$ LQG heat trace **in D.** Then

$$\mathbb{E}[\mathbf{H}(t)] = c_\gamma \mathbb{E}(M(D))\ t^{-1} - t^{-1+b(\gamma)+o(1)},\ \text{where } b(\gamma) = \frac{1}{2} + \frac{2}{\gamma^2}\left(\sqrt{1+\frac{\gamma^4}{16}} - 1\right).$$

- We **conjecture** that the analogous expansion holds for $\mathbf{N}(\lambda)$ as $\lambda \to \infty$.

Anomalous heat trace expansion

- $$\mathbf{H}(t) = \int_D \mathbf{p}_t^D(x,x)M(\mathrm{d}x)$$
  $$= \int_D \mathbf{p}_t^{\mathbb{C}}(x,x)M(\mathrm{d}x) - \int_D \mathbf{p}_t^{\mathbb{C}\setminus D}(x,x)M(\mathrm{d}x)$$

- (1) exact scale invariance of the heat kernel on Sheffield's **quantum cone**
  —> first term must correspond to area term
  (2) second term comes from points near boundary.

- Surprisingly, dominant behaviour comes from point with **atypical thickness,**
  $$\alpha = Q - \sqrt{Q^2 - 2}, \text{ where } Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

- A key point — **concentration** of exit time of a small ball with given circle average.

KPZ scaling for heat trace ?

- How can this be intrinsic?
- **Conjecture** (B.-Klein, '25+)
  With high probability $\mathbf{H}(t) = c_\gamma M(D)\ t^{-1} - t^{-1+\Delta+o(1)}$,
  where $\Delta \in [0,1]$, = **quantum scaling** exponent of ∂D.
- (Intuitively, $\dim_\gamma(\partial D)/\dim_\gamma(D) = 1 - \Delta$). The **KPZ scaling relation** (Duplantier-Sheffield, B.-Garban-Rhodes-Vargas, Gwynne-Holden-Miller):
  $$x = \frac{\gamma^2}{4}\Delta^2 + (1 - \frac{\gamma^4}{4})\Delta$$
  where $x \in [0,1]$, = Euclidean scaling exponent.
- The KPZ equation relates the ``random and deterministic dimensions'' of a set.

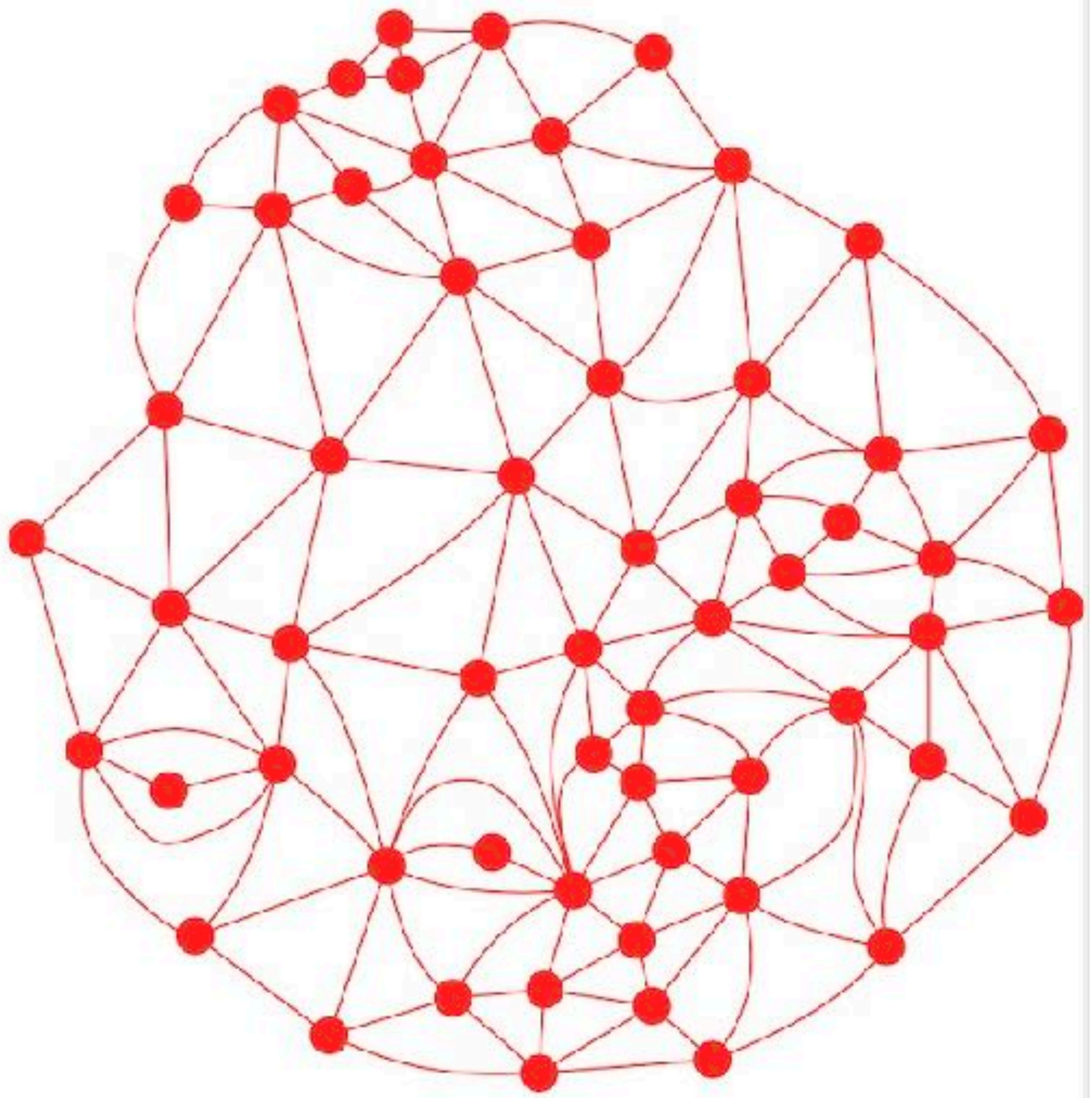
KPZ on quantum discs

- So this exponent depends sensitively on the boundary conditions.
- Most important case: **quantum disc = scaling limit of planar maps with disc topology.**

  Or, via work of Duplantier-Miller-Sheffield, boundary $\approx \mathrm{SLE}_\kappa$, with $\kappa = \gamma^2 \in (0,4)$.

- Then $\dim(\partial D) = 1 + \kappa/8$, so $x = 1 - (1+\kappa/8)/2 = 1/2 - \kappa/16$. (**Rohde—Schramm, Beffara**)
- Plug into KPZ and get $\Delta = 1/2$, indep. of γ.
- **Theorem** (B.-Klein, '25+). Now suppose (D, ϕ) is a quantum disc. Then

$$\mathbb{E}[\mathbf{H}(t)] = c_\gamma \mathbb{E}(M(D))\ t^{-1} - t^{-1/2+o(1)}.$$



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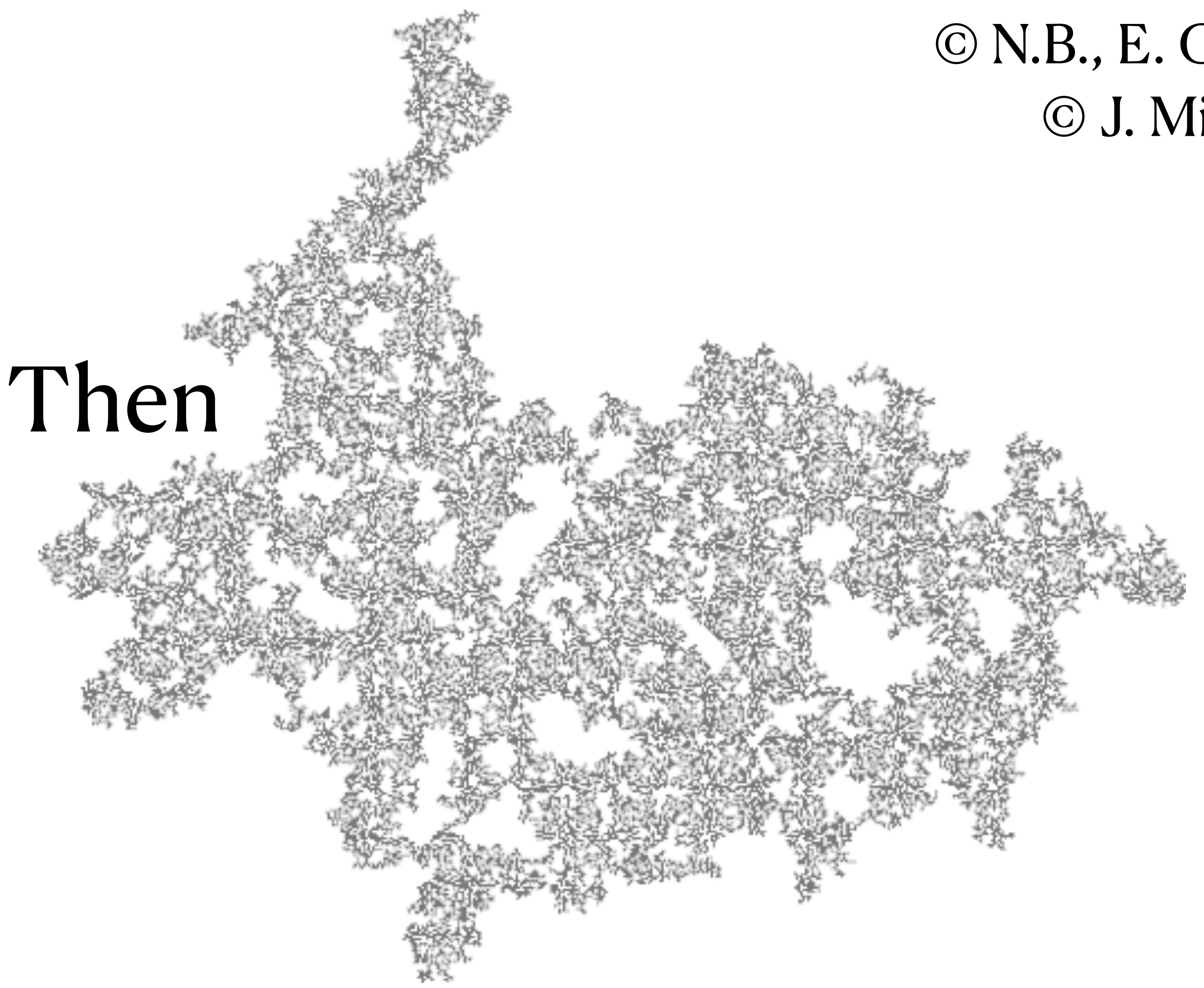


Heuristics for KPZ conjecture

- Let d_γ denote the Hausdorff dimension of the metric space associated to LQG_γ.
- (The value of d_γ is unknown even heuristically, except for $\gamma = \sqrt{8/3}$. Cf. Watabiki's prediction and work of **Ding-Gwynne**, **Gwynne-Pfeffer**, and **Budd**).
- It is expected that $\mathrm{dist}_\gamma(\mathbf{Z}_t, \mathbf{Z}_0) \approx t^{1/d_\gamma}$ for $t \to 0$.
- Cover the boundary with balls of radius $r = t^{1/d_\gamma}$ (need $N = r^{-\dim_\gamma(\partial D)} = t^{-(1-\Delta)}$ such balls).
- For points in such balls, we expect $\mathbf{p}_t^{\mathbb{C}\setminus D}(x,x) \approx 1/t$, otherwise ≈ 0.
- Hence $\displaystyle\int_D \mathbf{p}_t^{\mathbb{C}\setminus D}(x,x)\mu(dx) \approx N \times \mu(B(r)) \times 1/t = t^{-(1-\Delta)} \times r^{d_\gamma} \times (1/t) = t^{-(1-\Delta)}$.

Some natural questions

- **Quantum boundary length**, by renormalizing the heat content/heat trace?
- Any **estimate** on $|\mathbf{N}(\lambda) - c_\gamma M(D)\lambda|$?
- Can we get **zeta-regularized determinant**?
- **Polyakov-Alvarez** conformal anomaly?
- **Logarithmic fluctuations** of heat kernel?
- **Critical case** $\gamma = 2$?
- **Berezin-Li-Yau** inequalities?
- **Selberg**'s 1/4 conjecture?

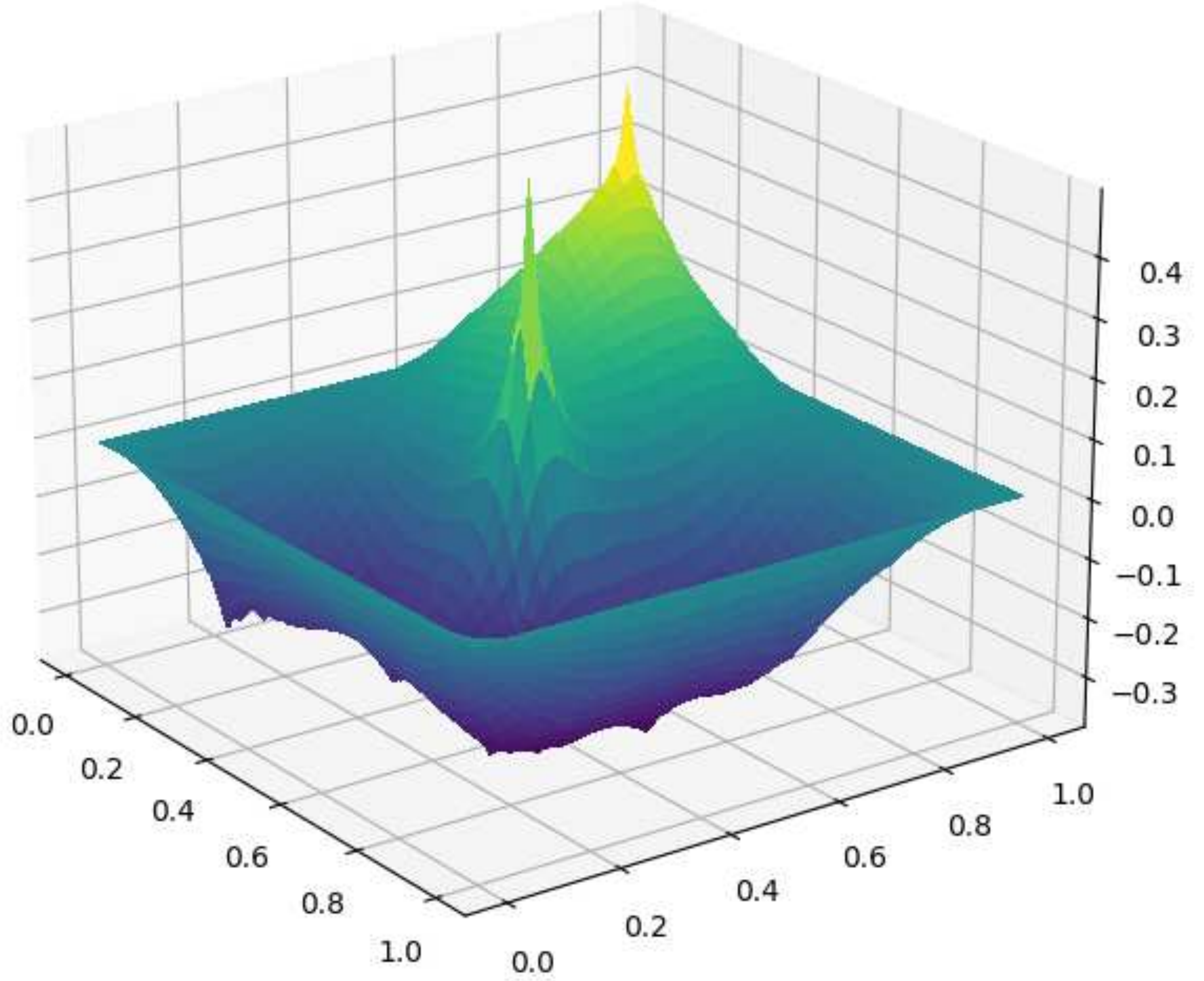
V. Conjectures.
Hearing the shape of LQG

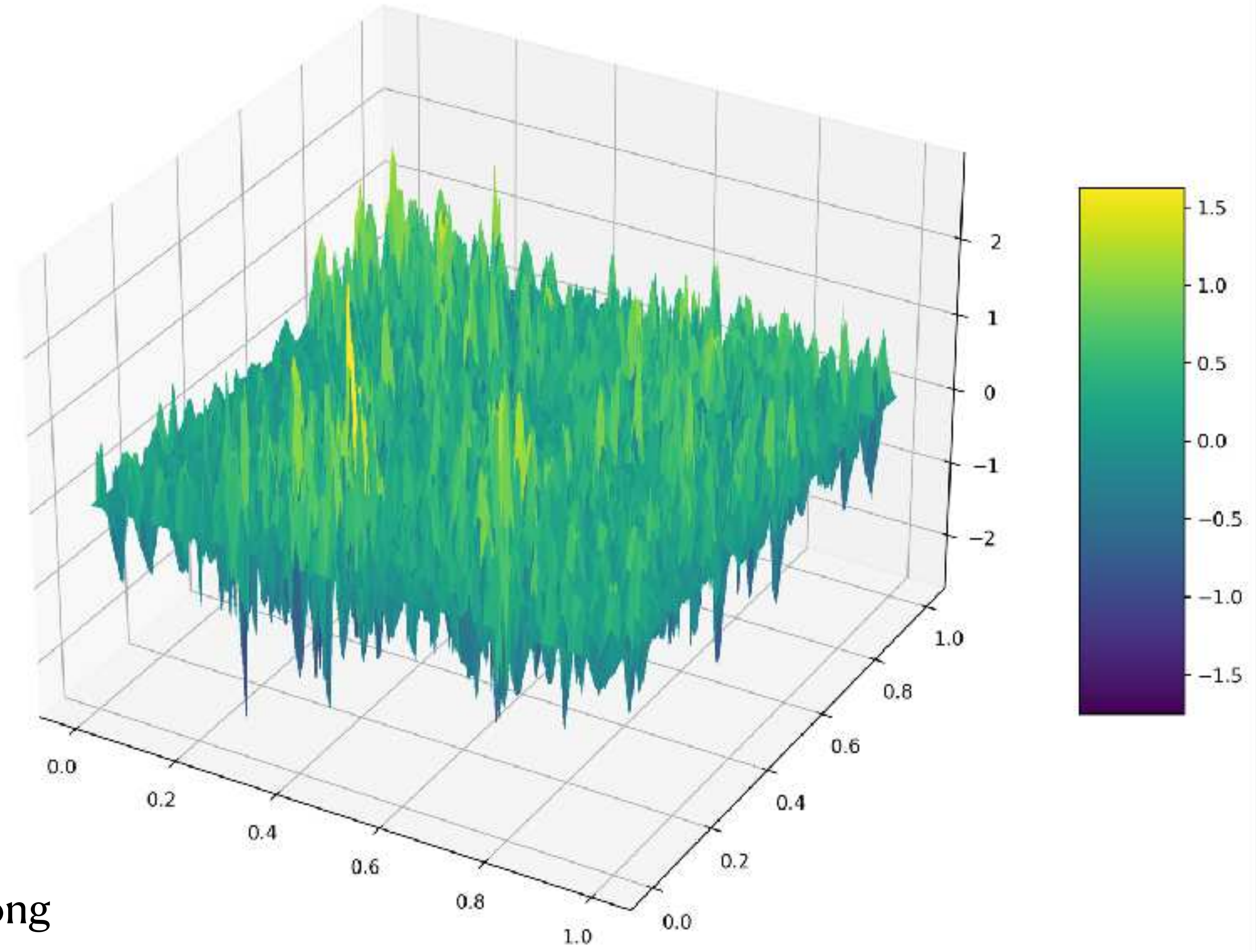
- **Conjecture (B.-Wong '23):**
  One CAN hear the shape of LQG (!)
  That is, ϕ is a.s. equal to a measurable function of the eigenvalues $\{\lambda_n\}_{n\geq 1}$.

- In fact, we conjecture that $\{\lambda_n\}_{n\geq 1}$ determines (D, ϕ) up to equivalence of random surfaces (Duplantier-Sheffield 2010)

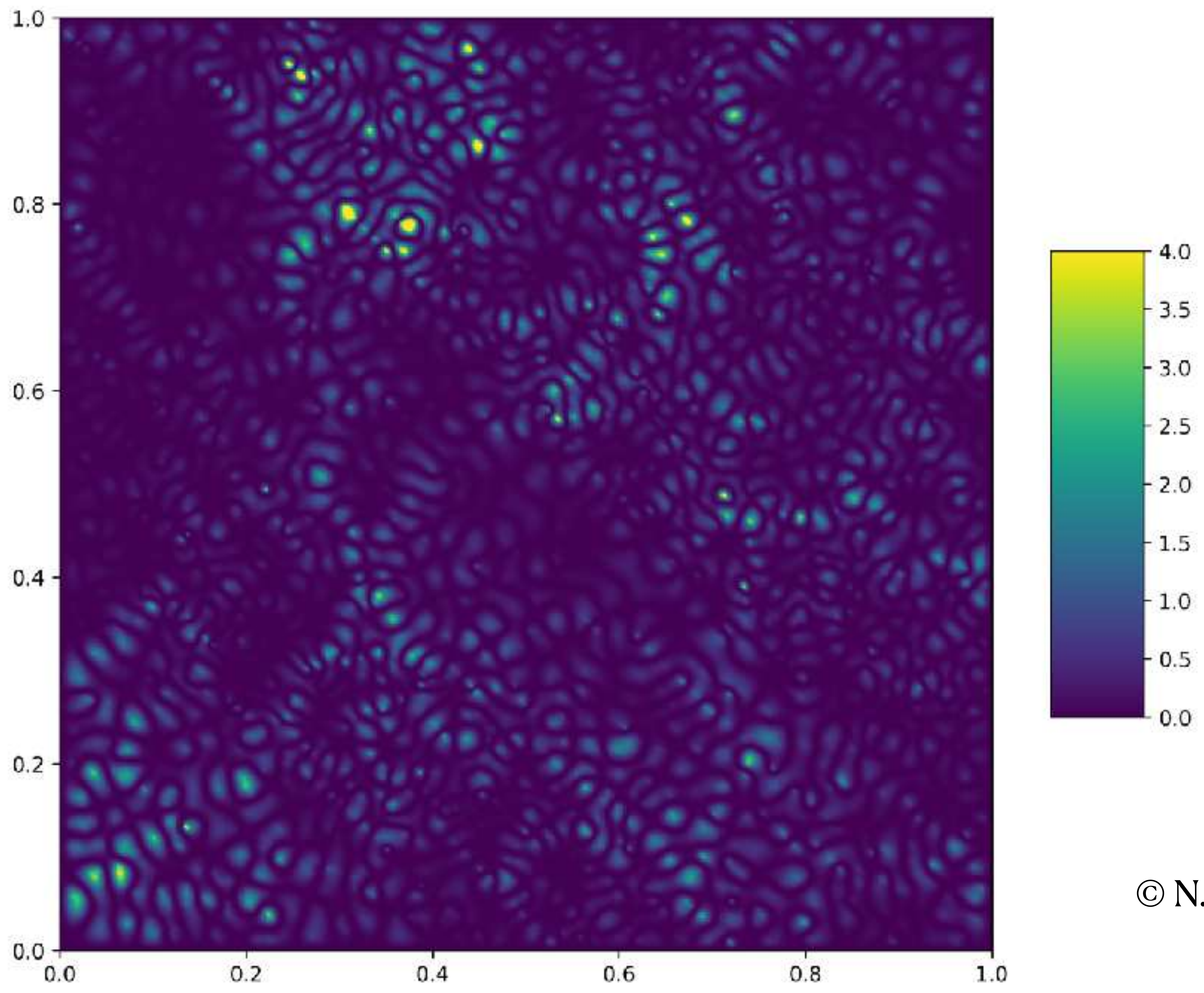
Localisation/Delocalisation

- Should eigenfunctions be delocalised (as for standard BM) or localised (as for Anderson model)?





Quantum Chaos



Conjecture: Eigenfunctions delocalized,

$$|\mathbf{f}_n(x)|^2 M(\mathrm{d}x) \Rightarrow M(\mathrm{d}x)$$

Motivation: connection to **quantum chaos!**

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Quantum Chaos

- Quantum chaos is manifestation at quantum level of ergodicity of **geodesic flow**:

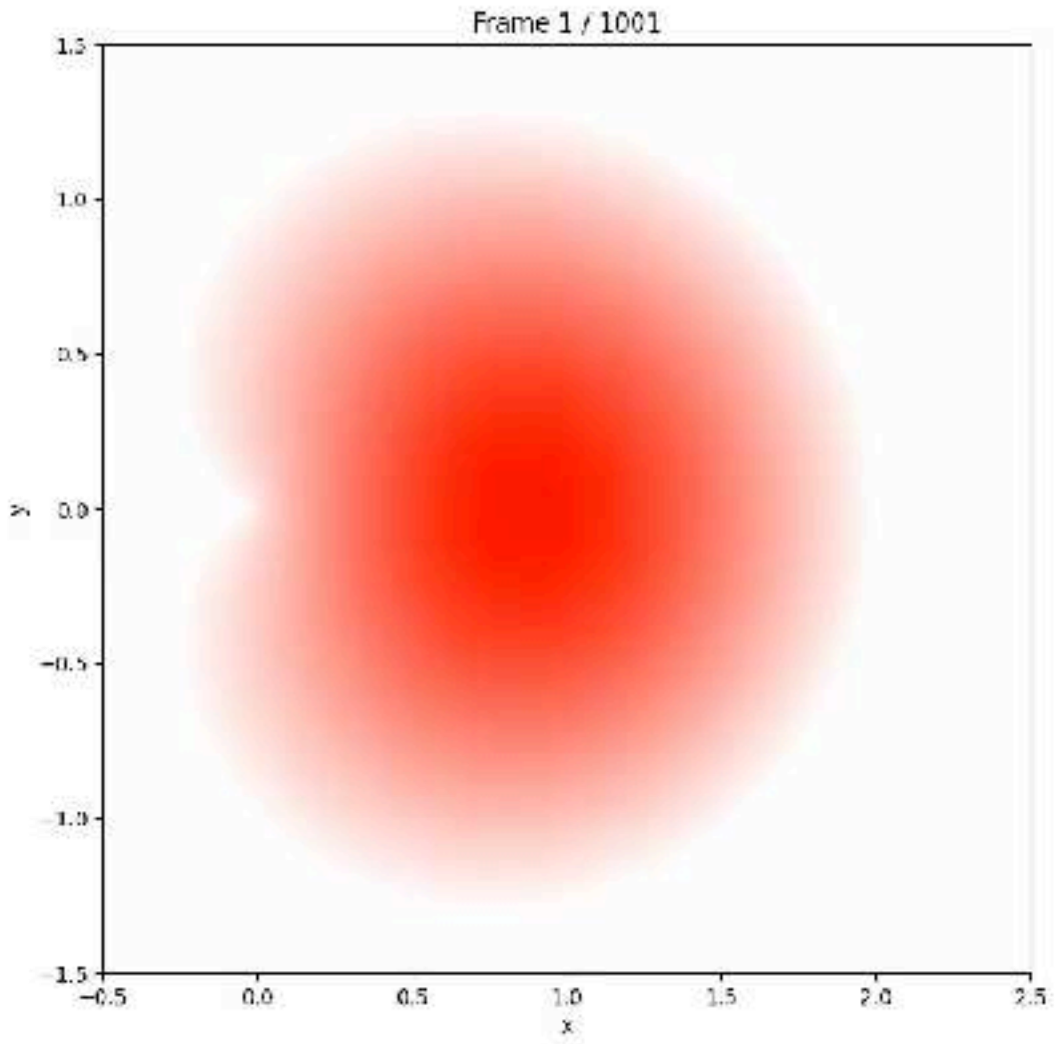
$$|f_n(x)|^2 \Rightarrow \nu_g(\mathrm{d}x), \quad (n \to \infty).$$

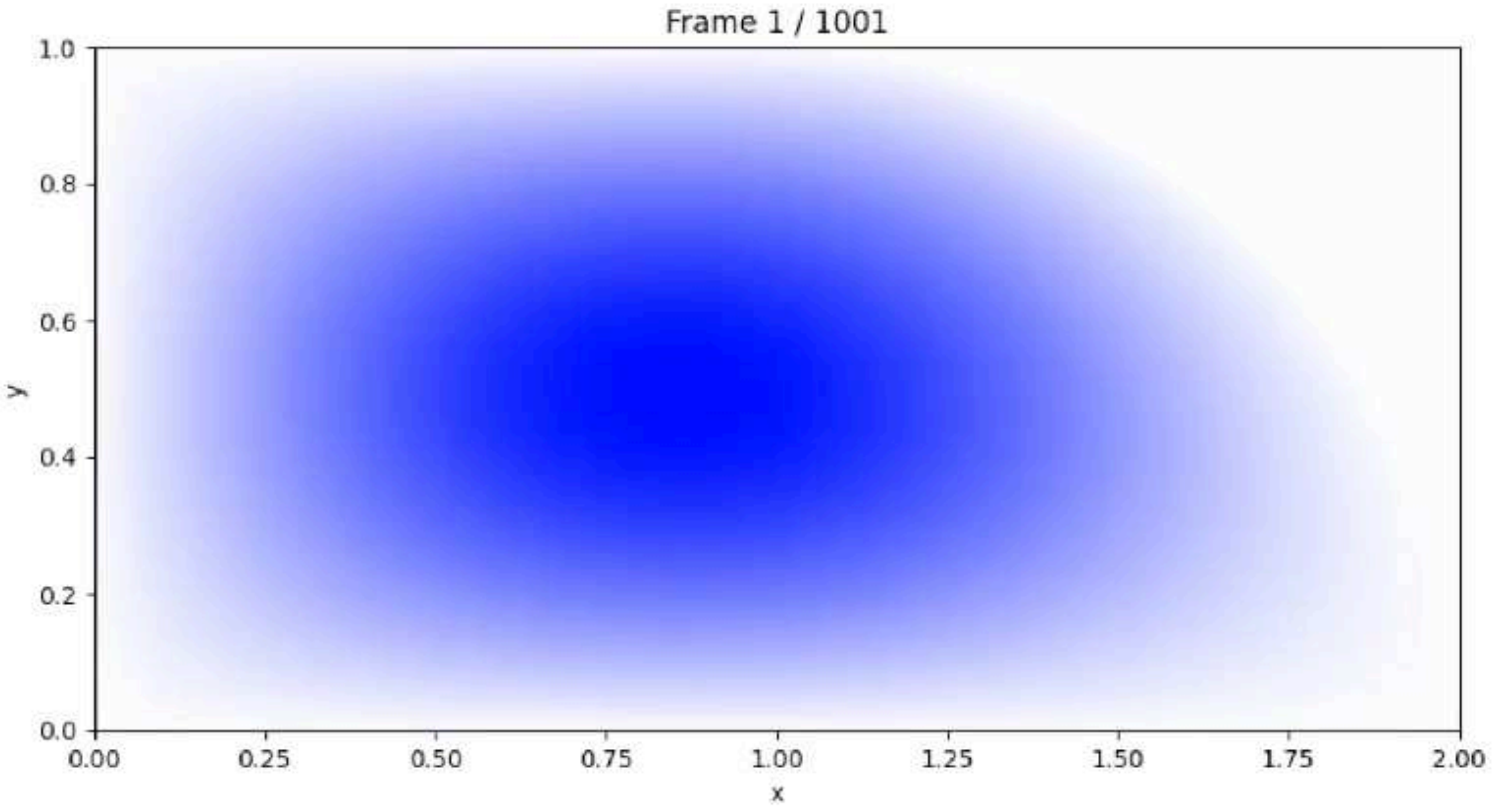
- Proved for **hyperbolic surfaces** up to dense subsequence $\{n_k\}_{k\geq 1}$ (Shnirelman 1974, Zelditch 1987, Colin de Verdière 1985).

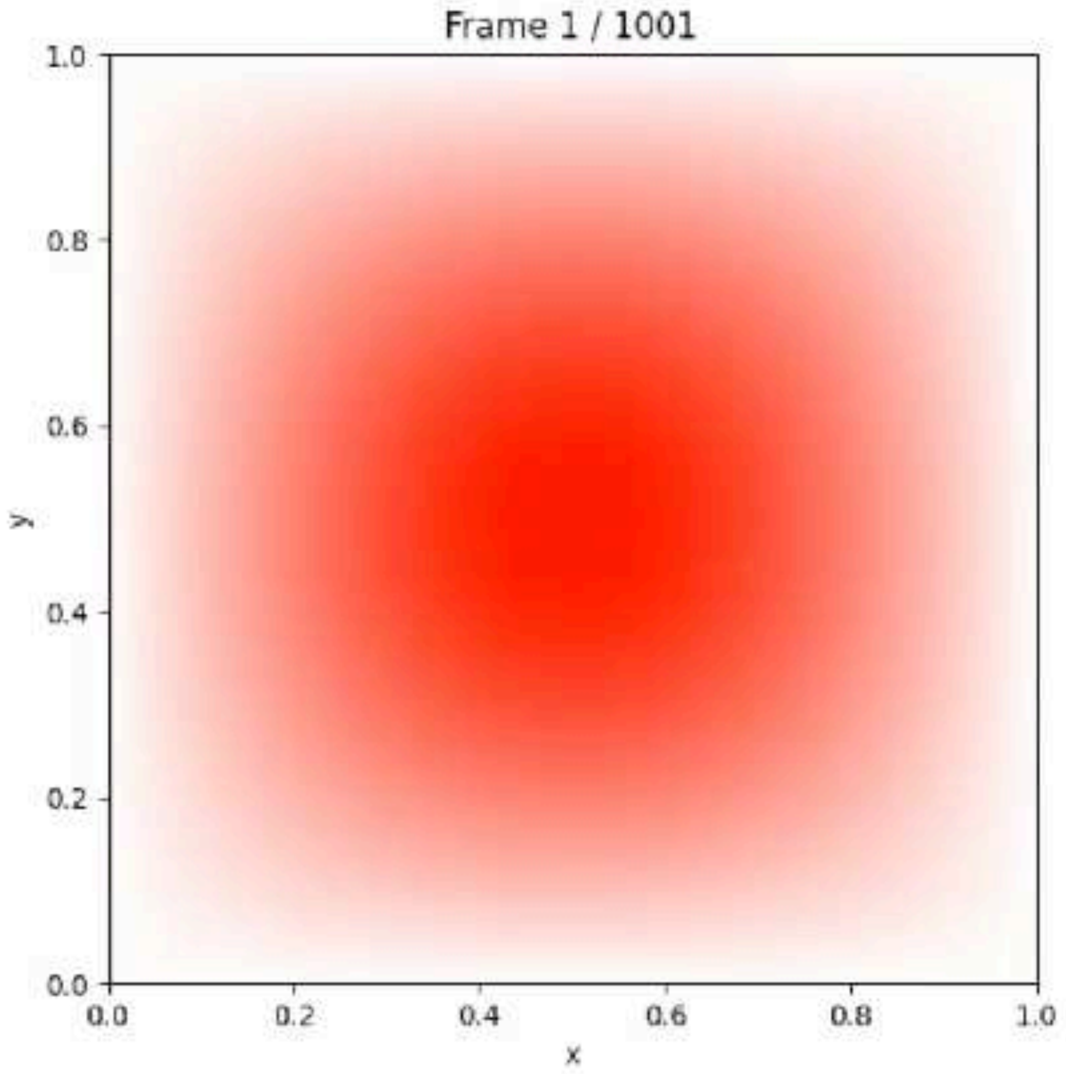
- Rudnick-Sarnak (1994): conjectured quantum ***unique*** ergodicity; Lindenstrauss (2006) for **arithmetic** surfaces.

- Polyakov's action minimized for ϕ of constant (negative) curvature → **hyperbolic** to 1st order! (Lacoin-Rhodes-Vargas 2020).

Random Waves in quantum chaos

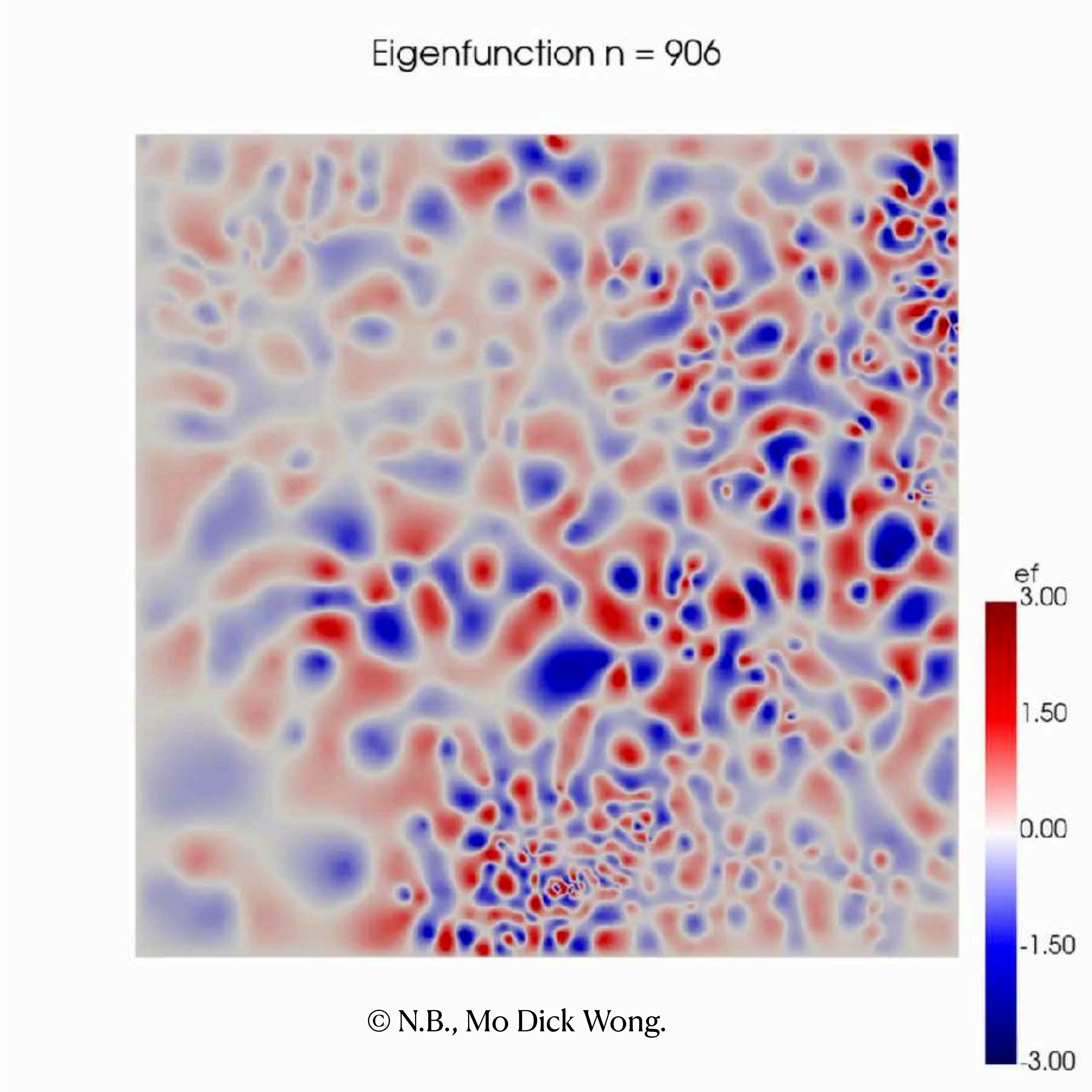






- The local behaviour of eigenfunction was predicted to converge to **Berry's random wave** model:
- Gaussian random field in plane with $\mathbb{E}[b(x)b(y)] = J_0(\|x - y\|)$, where J_0 = Bessel function of 1st kind

Random waves in LQG ?



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- Notice the ``filament structure''

- We also conjecture that LQG eigenfunctions converge locally to Berry's random wave (up to local scale change?)

Eigenvalue Spacing

For chaotic systems, quantum chaos also predicts eigenvalue repulsion.

- **Conjecture**:

  EV spacing $\xrightarrow[N\to\infty]{p} F_{\text{GOE}}(x)$,

  (this should also be true e.g. for planar maps !)