LECTURE 7

Glauber dynamics on a torus.

Critical droplet and metastable crossover time.

§ TARGET

In this lecture we analyse metastability for the Ising model in two and three dimensions subject to Glauber dynamics.

Spins live on a finite torus, flip up and down, want to align when they sit next to each other, and want to align with an external magnetic field. We are interested in how the system magnetises, i.e., how the dynamics aligns the spins with the magnetic field when initially all the spins are pointing in the opposite direction. Our goal will be to prove hypotheses (H1–H2) in Lecture 5, implying that THEOREMS 5.4–5.6 are valid.

In both two and three dimensions we will identify $(\Gamma^\star, \mathcal{C}^\star, K)$.

§ MODEL

1. Let $\Lambda \subset \mathbb{Z}^2$ be a large square torus, centred at the origin. With each vertex $x \in \Lambda$ we associate a spin variable $\sigma(x) \in \{-1, +1\}$, indicating whether the spin at x is pointing down or up. A configuration is denoted by $\sigma \in S = \{-1, +1\}^\Lambda$.

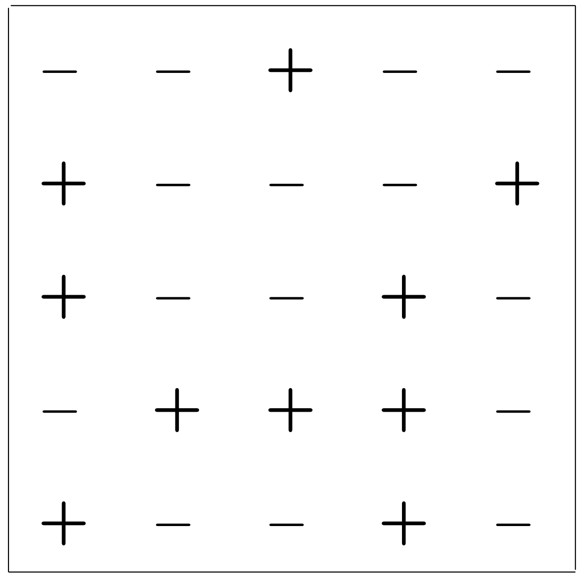
Each $\sigma \in S$ has an energy given by the Hamiltonian

$$H(\sigma) = -\frac{J}{2} \sum_{\{x,y\} \in \Lambda^*} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \Lambda} \sigma(x),$$

where

$$\Lambda^* = \Big\{ \{x, y\} \colon \ x, y \in \Lambda, \ \|x - y\| = 1 \Big\}$$

is the set of non-oriented nearest-neighbour edges in Λ. The interaction consists of a ferromagnetic pair potential $J > 0$ for each pair of nearest-neighbour spins in Λ and a magnetic field $h > 0$ for each spin in Λ.



An Ising-spin configuration.

Glauber dynamics is the Metropolis dynamics with respect to H at inverse temperature β with single-spin flips as the allowed moves, i.e., $-1 \leftrightarrow +1$ at single sites in Λ.

The Gibbs measure μ_β is the reversible equilibrium on S of the Metropolis dynamics with transition rates c_β on $S \times S$ (recall Lecture 5).

2. Throughout the sequel we assume that

$$h \in (0, 2J).$$

It will turn out that this parameter range corresponds to metastable behaviour in the limit as $\beta \to \infty$. A key role will be played by what we call the critical droplet size:

$$\ell_c = \left\lceil \frac{2J}{h} \right\rceil$$

($\lceil \cdot \rceil$ denotes the upper integer part). For reasons that will become clear later on, we will assume that

$$\frac{2J}{h} \notin \mathbb{N}.$$

Thus, an $(\ell_c - 1) \times (\ell_c - 1)$ droplet will be subcritical while an $\ell_c \times \ell_c$ droplet will be supercritical.

Moreover, we will assume that Λ is large enough so that it contains an $2\ell_c \times 2\ell_c$ square, which is necessary for (H1).

DEFINITION 7.1 Geometry of droplets

(a) Let

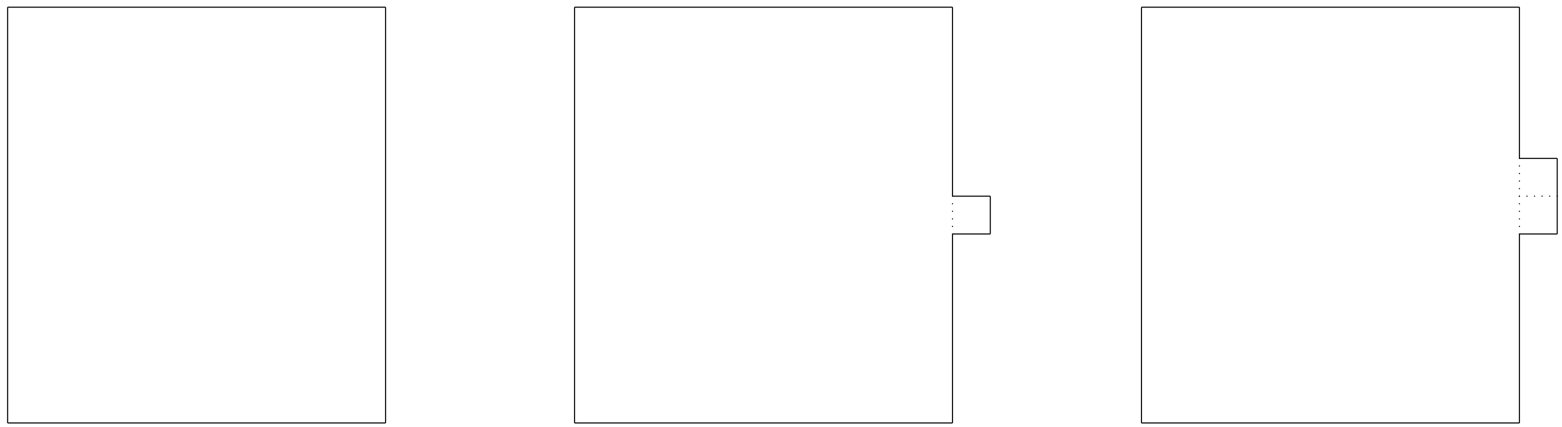
$$\boxminus = \{\sigma \in S\colon\ \sigma(x) = -1\ \forall\, x \in \Lambda\},$$
$$\boxplus = \{\sigma \in S\colon\ \sigma(x) = +1\ \forall\, x \in \Lambda\},$$

denote the configurations where all spins in Λ are down, respectively, up.

(b) Let $\mathcal{Q}$ be the set of configurations where the up-spins form a single $(\ell_c - 1) \times \ell_c$ quasi-square anywhere in Λ.

(c) Let $\mathcal{Q}^{1\mathrm{pr}}$ be the set of configurations where the up-spins form a single quasi-square $(\ell_c - 1) \times \ell_c$ anywhere in Λ with a single protuberance attached anywhere to one of its longest sides.

(d) Let $\mathcal{Q}^{2\mathrm{pr}}$ be the set of configurations where the up-spins form a single quasi-square $(\ell_c - 1) \times \ell_c$ anywhere in Λ with a double protuberance attached anywhere to one of its longest sides.



Configurations in $\mathcal{Q}$, $\mathcal{Q}^{1\text{pr}}$, $\mathcal{Q}^{2\text{pr}}$. The quasi-square has size $\ell_c \times (\ell_c - 1)$.
The up-spins sit inside the contours, the down-spins sit outside.

§ MAIN THEOREMS

THEOREM 7.2

The pair $(\boxminus, \boxplus)$ *satisfies hypotheses* (H1–H2) *in Lecture 5 and hence THEOREMS 5.4–5.6 hold.*

THEOREM 7.3

The pair $(\boxminus, \boxplus)$ *has protocritical set* $\mathcal{P}^\star(\boxminus, \boxplus) = \mathcal{Q}$, *critical set* $\mathcal{C}^\star(\boxminus, \boxplus) = \mathcal{Q}^{\mathrm{1pr}}$, *and communication height*

$$\Gamma^\star = \Gamma^\star(\boxminus, \boxplus) = H(\mathcal{Q}^{\mathrm{1pr}}) - H(\boxminus) = J[4\ell_c] - h[\ell_c(\ell_c - 1) + 1].$$

THEOREM 7.4

The prefactor $K = K(\Lambda)$ *equals* $K(\Lambda) = \frac{3}{4(2\ell_c - 1)} \frac{1}{|\Lambda|}$.

In addition, we have the following geometric description of the configurations in the valleys $S_{\boxminus}, S_{\boxplus}$ around $\boxminus, \boxplus$. Let

$$V_{\leq \mathcal{Q}} = \{\sigma \in S \colon \sigma \leq \sigma' \text{ for some } \sigma' \in \mathcal{Q}\},$$
$$V_{\geq \mathcal{Q}^{\text{2pr}}} = \{\sigma \in S \colon \sigma \geq \sigma' \text{ for some } \sigma' \in \mathcal{Q}^{\text{2pr}}\},$$

where we write $\sigma \leq \sigma'$ when $\sigma(x) \leq \sigma'(x)$ for all $x \in \Lambda$, and vice versa.

THEOREM 7.5

$$S_{\boxminus} \supseteq V_{\leq \mathcal{Q}},$$
$$S_{\boxplus} \supseteq V_{\geq \mathcal{Q}^{\text{2pr}}}.$$

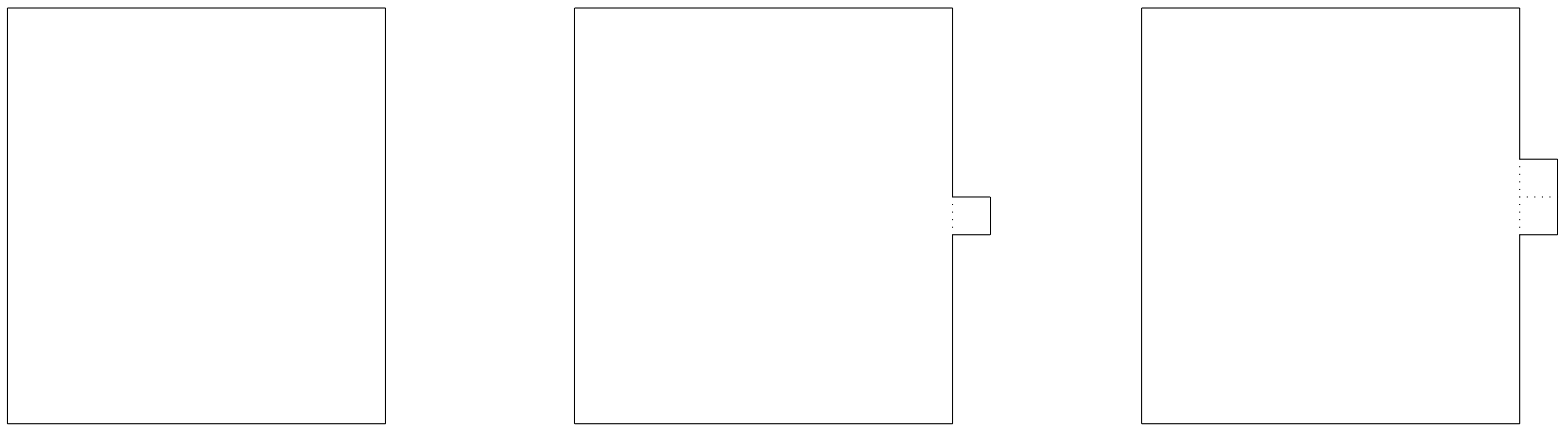
§ HEURISTICS

1. (H2) is easy to check, (H1) is more involved and relies on certain isoperimetric inequalities.

2. We will see that $\mathcal{Q}^{1\mathrm{pr}} \subseteq \mathcal{S}(\boxminus, \boxplus)$, the communication level set of the pair $(\boxminus, \boxplus)$. We will also see that on its way from $\boxminus$ to $\boxplus$ the dynamics passes through $\mathcal{S}(\boxminus, \boxplus)$ in three steps:

- Create a quasi-square of up-spins.
- Attach a single protuberance.
- Turn the latter into a double protuberance.

After these three steps are completed, the dynamics is over the hill and proceeds downwards in energy to fill the box with up-spins.



Transitions over the hill: $\mathcal{Q} \longrightarrow \mathcal{Q}^{\mathrm{1pr}} \longrightarrow \mathcal{Q}^{\mathrm{2pr}}$.

3. The average time it takes for the dynamics to enter $\mathcal{C}^\star(\boxminus, \boxplus) = \mathcal{Q}^{1\mathrm{pr}}$ when starting from $\boxminus$ is

$$\frac{1}{|\mathcal{Q}^{1\mathrm{pr}}|} \mathrm{e}^{\beta\Gamma^\star} [1 + o(1)], \qquad \beta \to \infty,$$

where $|\mathcal{Q}^{1\mathrm{pr}}|$ counts the number of critical droplets. Let $\pi(\ell_c)$ be the probability that the single protuberance turns into a double protuberance rather than disappears. Then

$$\frac{1}{\pi(\ell_c)} [1 + o(1)], \qquad \beta \to \infty,$$

is the average number of times a critical droplet attempts to move over the hill before it finally manages to do so. The average nucleation time is the product of the two, and so we conclude that

$$K = \frac{1}{|\mathcal{Q}^{1\mathrm{pr}}|\, \pi(\ell_c)}.$$

4. To compute $|\mathcal{Q}^{1\text{pr}}|$, note that

$$|\mathcal{Q}^{1\text{pr}}| = |\Lambda|\, N(\ell_c) \quad \text{with} \quad N(\ell_c) = 4\ell_c.$$

Indeed, the $(\ell_c - 1) \times \ell_c$ quasi-square is located anywhere in Λ (which is a torus) in two possible orientations, while the single protuberance is attached to any of the $2\ell_c$ possible locations on one of the sides of length ℓ_c.

5. To compute $\pi(\ell_c)$, note that if the protuberance sits at one of the two extreme ends of the side it is attached to, then the probability is $\frac{1}{2}$ that its one neighbouring spin flips up before before the spin itself flips down. On the other hand, if the protuberance sits somewhere else, then the probability is $\frac{2}{3}$ that one of its two neighbouring spins on the same side flip up before the spin itself flips down.

The location of the protuberance is uniform, because of the uniform exit distribution stated in THEOREM 5.4(b). Therefore we get

$$\pi(\ell_c) = \frac{1}{\ell_c}\left\{2\,\frac{1}{2} + (\ell_c - 2)\,\frac{2}{3}\right\} = \frac{2\ell_c - 1}{3\ell_c}.$$

Combine the above observations to get the formula for K in THEOREM 7.4.

§ GEOMETRIC DEFINITIONS

In order to prove THEOREMS 7.2–7.5, we need some further definitions.

1. Throughout the sequel, we identify a configuration $\sigma \in S$ with the set of locations of the up-spins $\text{supp}(\sigma) = \{x \in \Lambda\colon \sigma(x) = +1\}$, and write $x \in \sigma$ to indicate that σ has an up-spin at x.

2. Given a configuration $\sigma \in S$, consider the set $C(\sigma) \subseteq \mathbb{R}^2$ defined as the union of the closed unit squares centred at the sites of $\text{supp}(\sigma)$. The maximal connected components $C_1, \ldots, C_m$, $m \in \mathbb{N}$, of $C(\sigma)$ are called clusters of σ (two unit squares touching only at the corners are not connected). There is a 1-to-1 correspondence between configurations $\sigma \in S$ and sets $C(\sigma)$.

3. For $\sigma \in S$, let $|\sigma|$ be the volume of $C(\sigma)$, $\partial(\sigma)$ the boundary of $C(\sigma)$, called the contour of σ, and $|\partial(\sigma)|$ the length of $\partial(\sigma)$, i.e., the number of edges in Λ^* carrying opposite spins in σ. Then the energy associated with σ is given by

$$H(\sigma) = J|\partial(\sigma)| - h|\sigma| + H(\boxminus).$$

Note here the occurrence of a surface term and a volume term.

4. To describe the shape of clusters, we need the following:

- An $\ell_1 \times \ell_2$ rectangle is a union of closed unit squares centered at the sites in Λ with side lengths $\ell_1, \ell_2 \geq 1$. We use the convention $\ell_1 \leq \ell_2$ and collect rectangles in equivalence classes modulo translations and rotations.
- A quasi-square is an $\ell \times (\ell + \delta)$ rectangle with $\ell \geq 1$ and $\delta \in \{0, 1\}$. A square is a quasi-square with $\delta = 0$.
- A bar is a $1 \times k$ rectangle with $k \geq 1$. A bar is called a row or a column if it fills a side of a rectangle.
- A corner of a rectangle is an intersection of two bars attached to the rectangle.
- A 1-protuberance is a 1×1 rectangle attached to one side of a rectangle.
- A 2-protuberance is a 1×2 rectangle attached to one side of a rectangle.

5. The configuration space S can be partitioned as

$$S = \bigcup_{n=0}^{|\Lambda|} \mathcal{V}_n,$$

where

$$\mathcal{V}_n = \{\sigma \in S \colon \ |\sigma| = n\}$$

is the set of configurations with n up-spins.

§ VERIFICATION OF THE TWO HYPOTHESES

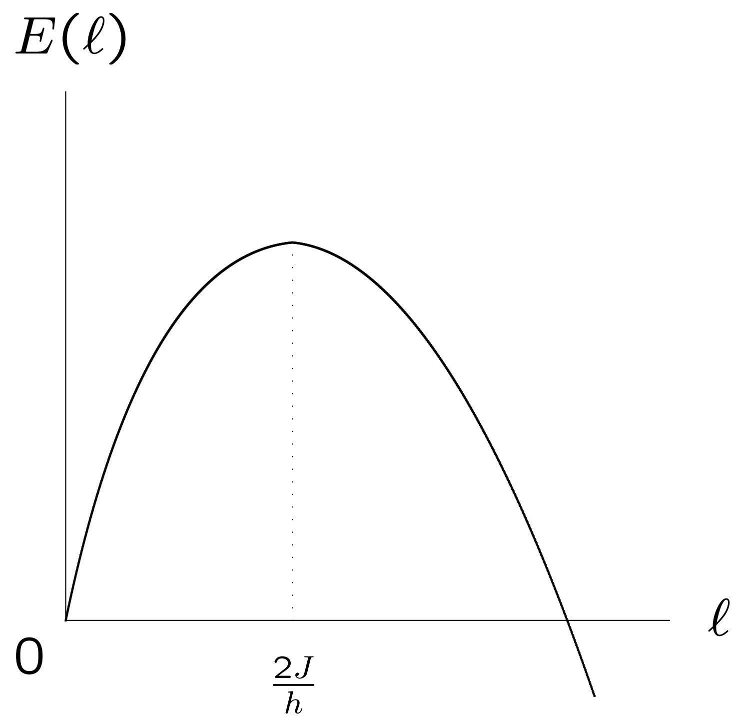
▶ (H1)

1. Let D_ℓ denote the set of configurations where the up-spins form a single $\ell \times \ell$ square anywhere in Λ. The energy of the configurations in D_ℓ equals

$$E(\ell) = H(D_\ell) - H(\boxminus) = J[4\ell] - h[\ell]^2,$$

which is maximal at $\ell = 2J/h$ and is negative for $\ell > 4J/h$.

2. Since Λ is chosen large enough so that it contains a $2\ell_c \times 2\ell_c$ square, it follows that $H(\boxminus) = H(0 \times 0) > H(\boxplus)$. It is obvious that $\boxplus$ is the global minimum of H, while $\boxminus$ is a local minimum of H. Thus, to settle (H1) it remains to show that $\boxminus$ has the unique maximal stability level on $S \backslash \boxplus$.



$$\ell \mapsto E(\ell).$$

3. Let $\gamma^\star = (\gamma_0^\star, \dots, \gamma_{|\Lambda|}^\star)\colon \boxminus \to \boxplus$ be any path that grows a droplet of up-spins by successively adding rows and bars to a quasi-square or square. We refer to this as the reference path. Below we will show that

$$\gamma^\star \in (\boxminus \to \boxplus)_{\text{opt}},$$
$$H(\gamma_k^\star) = \min_{\sigma \in \mathcal{V}_k} H(\sigma).$$

Let

$$k^\star = \min\{k \in \mathbb{N}\colon\ H(\gamma_k^\star) \leq H(\boxminus)\}$$

be the first time the reference path after it has left $\boxminus$ hits an energy not exceeding that of $\boxminus$.

4. For $\sigma, \sigma' \in S$, let $\sigma \vee \sigma'$ and $\sigma \wedge \sigma'$ be the componentwise maximum, respectively, minimum of σ and σ'. An easy computation shows that, for all $\sigma, \sigma' \in S$,

$$|\partial(\sigma \vee \sigma')| + |\partial(\sigma \wedge \sigma')| \leq |\partial(\sigma)| + |\partial(\sigma')|,$$
$$|\sigma \vee \sigma'| + |\sigma \wedge \sigma'| = |\sigma| + |\sigma'|.$$

Pick any $\sigma \in S \setminus [\boxminus \cup \boxplus]$. Then there exists at least one pair of neighbouring sites x and y in Λ such that $\sigma(x) = -1$ and $\sigma(y) = +1$. By translation invariance we may assume without loss of generality that the first two spins that are flipped up in $\gamma^\star$ are located at x and y, respectively. Then

$$\sigma \wedge \gamma_1^\star = \boxminus,$$
$$1 \leq |\sigma \wedge \gamma_k^\star| < k \quad \forall\, k \geq 2.$$

5. In what follows we consider the σ-augmentation of the reference path given by $\sigma \vee \gamma_k^\star$ for $0 \leq k \leq k^\star$, and show:

For any $\sigma \in S \setminus [\boxminus \cup \boxplus]$, the σ-augmentation of the reference path faces a lower hill than the reference path, due to the attractive interaction.

We have

$$H(\sigma \vee \gamma_1^\star) - H(\sigma) < H(\boxminus \vee \gamma_1^\star) - H(\boxminus) = H(\gamma_1^\star) - H(\boxminus).$$

Moreover, for $2 \leq k \leq k^\star$ we can estimate

$$\begin{aligned} &H(\sigma \vee \gamma_k^\star) - H(\sigma) \\ &= J\Big[|\partial(\sigma \vee \gamma_k^\star)| - |\partial(\sigma)|\Big] - h\Big[|\sigma \vee \gamma_k^\star| - |\sigma|\Big] \\ &\leq J[|\partial(\gamma_k^\star)| - |\partial(\sigma \wedge \gamma_k^\star)|] - h[|\gamma_k^\star| - |\sigma \wedge \gamma_k^\star|] \\ &= H(\gamma_k^\star) - H(\sigma \wedge \gamma_k^\star) \\ &< H(\gamma_k^\star) - H(\boxminus). \end{aligned}$$

(Note that $H(\sigma \wedge \gamma_k^\star) > H(\boxminus)$ for $2 \leq k \leq k^\star$.) By picking $k = k^\star$, we get

$$H(\sigma \vee \gamma_{k^\star}^\star) - H(\sigma) < H(\gamma_{k^\star}^\star) - H(\boxminus) \leq 0.$$

6. Combining, we find that

$$H(\sigma \vee \gamma_{k^\star}^\star) < H(\sigma),$$
$$\Phi(\sigma, \sigma \vee \gamma_{k^\star}^\star) - H(\sigma) < \max_{1 \leq k \leq k^\star} H(\gamma_k^\star) - H(\boxminus) \leq \Gamma^\star,$$

where the second inequality uses that $\sigma = \sigma \vee \gamma_0^\star$ with $\gamma_0^\star = \boxminus$ and the third inequality uses that $\gamma^\star \in (\boxminus \to \boxplus)_{\text{opt}}$. Because of DEFINITION 5.1(c), what this says is that the stability level of σ is $< \Gamma^\star$.

7. Since $\Phi(\boxminus, \boxplus) - H(\boxminus) = \Gamma^\star$, it follows that $\boxminus$ has the unique maximal stability level on $S \backslash \boxplus$ (recall LEMMA 5.7).

▶ (H2)

It follows from DEFINITIONS 7.1(b–c) and THEOREM 7.3 that (H2) is satisfied. Indeed, for each configuration in $\mathcal{C}^\star(\boxminus,\boxplus) = \mathcal{Q}^{\text{1pr}}$ there is exactly one configuration in $\mathcal{P}^\star(\boxminus,\boxplus) = \mathcal{Q}$ from which it can be reached via an allowed move, namely, the configuration obtained by removing the single protuberance.

§ STRUCTURE OF COMMUNICATION LEVEL SET

To prove THEOREMS 7.3 and THEOREM 7.5, argue as follows.

PROPOSITION 7.6
(i) $\Phi(\boxminus, \boxplus) = \Gamma^{\star}$.
(ii) $\mathcal{S}(\boxminus, \boxplus) \supseteq \mathcal{Q}^{1\text{pr}}$.

PROOF:

(i) The proof is based on four lemmas, stated in LEMMAS 7.7–7.10 below.

• $\Phi(\boxminus, \boxplus) \leq \Gamma^\star$:

All we need to do is to construct a path that connects $\boxminus$ and $\boxplus$ without exceeding energy $\Gamma^\star$. The proof comes in three steps.

1. We first show that the configurations in $\mathcal{Q}$ are connected to $\boxminus$ by a path that stays below $\Gamma^\star$.

LEMMA 7.7 *For any* $\sigma \in \mathcal{Q}$ *there exists a* $\gamma\colon \sigma \to \boxminus$ *such that* $\max_{\xi \in \omega} H(\xi) < \Gamma^\star$.

PROOF: Fix $\sigma \in \mathcal{Q}$. Note that

$$H(\sigma) = \Gamma^\star - (2J - h).$$

First, we flip down a spin at a corner of the quasi-square, which increases the energy by h.

Next, we repeat this operation another $\ell_c - 3$ times, each time picking a spin from a corner on the same shortest side. To guarantee that we never reach energy $\Gamma^\star$, we must have that

$$h(\ell_c - 2) < 2J - h,$$

or

$$\ell_c < \frac{2J}{h} + 1.$$

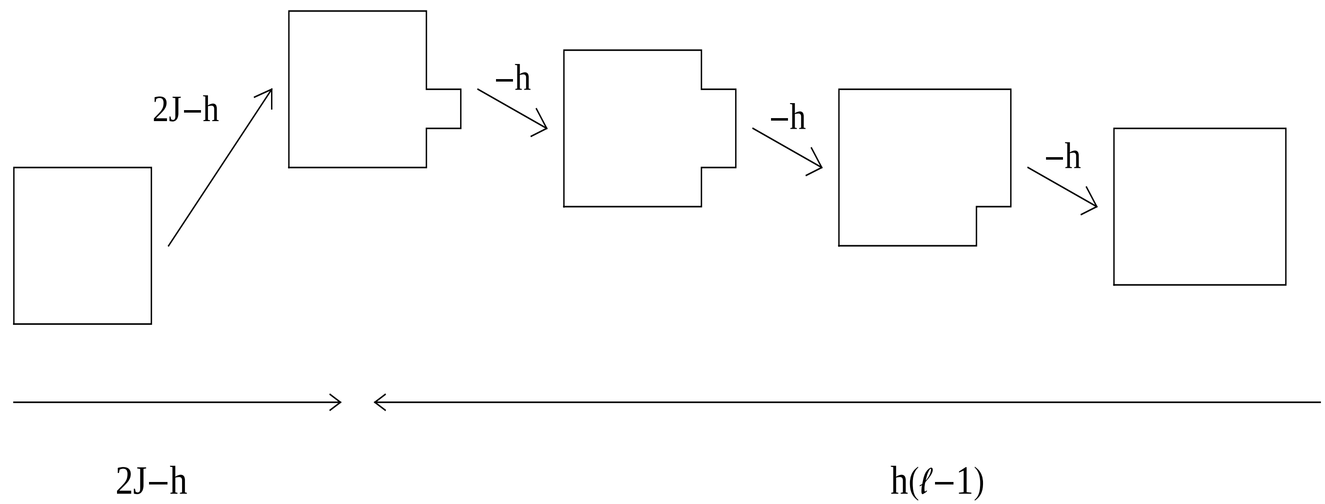
But this inequality holds by the definition of ℓ_c and the non-degeneracy hypothesis.

Finally, we flip down the last spin, which lowers the energy by $2J - h$, so that we arrive at energy

$$\Gamma^\star - (2J - h) - [2J - h(\ell_c - 1)],$$

which is strictly smaller than $\Gamma^\star - (2J - h)$. Thus, the removal of a row of length $\ell_c - 1$ from the $(\ell_c - 1) \times \ell_c$ quasi-square in σ lowers the energy.

We now have a square of side length $\ell_c - 1$. It is obvious that we can remove further rows without encountering any new conditions, until we reach $\boxminus$. $\square$



Cost of adding or removing a row of length ℓ.

2. We next show that configurations in $\mathcal{Q}^{2\text{pr}}$ are connected to $\boxplus$ by a path that stays below $\Gamma^\star$.

LEMMA 7.8 *For any* $\sigma \in \mathcal{Q}^{2\text{pr}}$ *there exists an* $\gamma\colon \sigma \to \boxplus$ *such that* $\max_{\xi \in \omega} H(\xi) < \Gamma^\star$.

PROOF: Fix $\sigma \in \mathcal{Q}^{2\text{pr}}$. Note that

$$H(\sigma) = \Gamma^\star - h.$$

First, we flip up a spin next to the 2-protuberance. This lowers the energy by h. We can repeat this operation another $\ell_c - 3$ times until the row is filled. By that time we have a square of side length ℓ_c and energy

$$\Gamma^\star - h(\ell_c - 1).$$

Next, we flip up a spin to form a new 1-protuberance. This raises the energy by $2J - h$. To make sure that we do not reach energy $\Gamma^\star$, we must have

$$h(\ell_c - 1) > 2J - h,$$

or

$$\ell_c > \frac{2J}{h},$$

which holds by the definition of ℓ_c and the non-degeneracy hypothesis. We now have a square of side length ℓ_c with a 1-protuberance.

By flipping up a spin next to this 1-protuberance, we get a 2-protuberance and reach energy

$$\Gamma^\star - h(\ell_c - 1) + (2J - h) - h.$$

This is strictly smaller than $\Gamma^* - h$. Thus, the completion of a row of length ℓ_c with a 2-protuberance and the creation of a new 2-protuberance lowers the energy.

It is obvious that we can complete further rows and create further 2-protuberances without encountering any new conditions, until we reach $\boxplus$. $\square$

3. The desired path $\gamma\colon \boxminus \to \boxplus$ is realised by tracing the path in LEMMA 7.7 in the reverse direction, from $\boxminus$ to $\sigma \in \mathcal{Q}$, then going from σ to $\sigma' \in \mathcal{Q}^{1\mathrm{pr}}$ by adding a 1-protuberance and from σ' to $\sigma'' \in \mathcal{Q}^{2\mathrm{pr}}$ by extending this 1-protuberance to a 2-protuberance, and finally following the path in LEMMA 7.8 from σ'' to $\boxplus$. This γ will be called the reference path for the magnetisation.

- $\Phi(\boxminus, \boxplus) \geq \Gamma^\star$:

4. The first key ingredient is the following observation:

LEMMA 7.9 *Any* $\omega \in (\boxminus \to \boxplus)_{\text{opt}}$ *must pass through* $\mathcal{Q}$.

PROOF: Any path $\gamma \colon \boxminus \to \boxplus$ must cross the set $\mathcal{V}_{\ell_c(\ell_c-1)}$.
The following isoperimetric inequality holds:

> In $\mathcal{V}_{\ell_c(\ell_c-1)}$ the unique (modulo translations and rotations) configuration of minimal energy is the $(\ell_c - 1) \times \ell_c$ quasi-square, which has energy $H(\sigma) = \Gamma^\star - (2J - h)$.

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All other configurations in $\mathcal{V}_{\ell_c(\ell_c-1)}$ have energy at least $\Gamma^\star + h$, and thus any path not hitting $\mathcal{Q}$ exceeds energy $\Gamma^\star$.
$\square$

5. The second key ingredient is the following observation:

LEMMA 7.10 *Any* $\gamma \in (\boxminus \to \boxplus)_{\text{opt}}$ *must pass through* $\mathcal{Q}^{1\text{pr}}$.

PROOF: Follow the path until it hits the set $\mathcal{V}_{\ell_c(\ell_c-1)}$. Then, according to LEMMA 7.9, the configuration in this set must be an $(\ell_c-1)\times\ell_c$ quasi-square. Since we need not consider paths that return to the set $\mathcal{V}_{\ell_c(\ell_c-1)}$ afterwards, a first step beyond the quasi-square must be the creation of a 1-protuberance. This brings us to energy $\Gamma^\star$.

If the 1-protuberance is created on the side of length ℓ_c, then we have a configuration in $\mathcal{Q}^{1\text{pr}}$. If, on the other hand, it is created on the side of length $\ell_c - 1$, then completion of the row leads an $(\ell_c-1)\times(\ell_c+1)$ rectangle with energy $\Gamma^\star - h(\ell_c-2)$.

After that the creation of a 1-protuberance brings us to energy $\Gamma^\star - h(\ell_c - 2) + (2J - h)$, which exceeds energy $\Gamma^\star$.

Since $(\ell_c - 1) \times (\ell_c + 1) + 1 = \ell_c \times \ell_c$, any other path that proceeds from the $(\ell_c - 1) \times \ell_c$ quasi-square with a 1-protuberance on the side of length $\ell_c - 1$ to the set $\mathcal{V}_{\ell_c^2}$ without returning to the set $\mathcal{V}_{\ell_c(\ell_c-1)}$ also exceeds energy $\Gamma^\star$. Indeed, the unique configuration with minimal energy in the set $\mathcal{V}_{\ell_c^2}$ is the $\ell_c \times \ell_c$ square (modulo rotations and translations). $\square$

LEMMAS 7.9–7.10 imply that $\Phi(\boxminus, \boxplus) \geq \Gamma^\star$, and with Steps 1–3 complete the proof of PROPOSITION 7.6 (i).

(ii) PROPOSITION 7.6(ii) follows from LEMMA 7.10 because $H(\mathcal{Q}^{\mathsf{1pr}}) = \Gamma^\star$. $\square$

The relations $\mathcal{P}^\star(\boxminus, \boxplus) = \mathcal{Q}$ and $\mathcal{C}^\star(\boxminus, \boxplus) = \mathcal{Q}^{\mathsf{1pr}}$ and the formula for $\Gamma^\star$ stated in THEOREM 7.3 are consequences of DEFINITION 5.3 and LEMMAS 7.7–7.10.

The claim in THEOREM 7.5 is immediate from LEMMAS 7.7–7.8 in combination with THEOREM 6.2, LEMMA 6.5, the formula for the harmonic function, and the earlier estimate $H(\sigma \vee \gamma_k^\star) - H(\sigma) < H(\gamma_k^\star) - H(\boxminus)$, $1 \le k \le k^\star$.

§ COMPUTATION OF THE PREFACTOR

We prove THEOREM 7.4.

PROOF: Our starting point is the variational formula for $\Theta = 1/K$ in LEMMA 6.7. This simplifies considerably because of the following two facts that are specific to our Glauber dynamics (abbreviate $\mathcal{C}^\star = \mathcal{C}^\star(\boxminus, \boxplus)$):

- $S^\star \setminus [S_\boxminus \cup S_\boxplus] = \mathcal{C}^\star$, i.e., there are no wells inside $\mathcal{C}^\star$.
- There are no allowed moves within $\mathcal{C}^\star$, i.e., critical droplets cannot transform into each other via single spin-flips.

Consequently,

$$\Theta = \min_{h:\ \mathcal{Q}^{\text{1pr}} \to [0,1]} \sum_{\sigma \in \mathcal{Q}^{\text{1pr}}} \Big\{ [1 - h(\sigma)]^2 \, N^-(\sigma) + [h(\sigma)]^2 \, N^+(\sigma) \Big\},$$

$$= \sum_{\sigma \in \mathcal{Q}^{\text{1pr}}} \frac{N^-(\sigma) N^+(\sigma)}{N^-(\sigma) + N^+(\sigma)},$$

where

$$\begin{aligned} N^-(\sigma) &= |\{\sigma' \in \mathcal{Q}\colon\ \sigma \sim \sigma'\}|, \\ N^+(\sigma) &= |\{\sigma' \in \mathcal{Q}^{\text{2pr}}\colon\ \sigma \sim \sigma'\}|, \end{aligned}$$

is the number of configurations in $\mathcal{Q}$, respectively, $\mathcal{Q}^{\text{2pr}}$ that can reached from $\sigma \in \mathcal{Q}^{\text{1pr}}$ by a single spin-flip (use that $\mathcal{Q} \subseteq S_{\boxminus}$ and $\mathcal{Q}^{\text{2pr}} \subseteq S_{\boxplus}$).

For all $\sigma \in \mathcal{Q}^{\text{1pr}}$ we have $N^{-}(\sigma) = 1$, $N^{+}(\sigma) = 1$ when the 1-protuberance in σ sits at a corner, and $N^{+}(\sigma) = 2$ when it does not. Hence

$$\Theta = 2|\Lambda|\,[2(\ell_c - 2)\,\tfrac{2}{3} + 4\,\tfrac{1}{2}] = |\Lambda|\,\tfrac{4}{3}(2\ell_c - 1),$$

where $2|\Lambda|$ counts the number of locations and rotations of the protocritical droplet. Since $K = 1/\Theta$, this completes the proof of THEOREM 7.4. □

§ EXTENSION TO THREE DIMENSIONS

We briefly indicate how to extend the main definitions and results from two to three dimensions. No proofs are given.

Let $\Lambda \subset \mathbb{Z}^3$ be a large cubic box, centred at the origin. The metastable parameter range is

$$h \in (0, 3J),$$

and we assume that

$$\frac{2J}{h} \notin \mathbb{N}, \qquad \frac{4J}{h} \notin \mathbb{N}.$$

The analogue of DEFINITION 7.1(b–c) reads as follows:

DEFINITION 7.12 Geometry of droplets

(a) Let $\mathcal{Q}$ be the set of configurations where the up-spins form an $(m_c-1)\times(m_c-\delta_c)\times m_c$ quasi-cube with, attached to one of its faces, an $(\ell_c-1)\times\ell_c$ quasi-square, anywhere in Λ. Here, $\delta_c\in\{0,1\}$ depends on the arithmetic properties of J and h, while

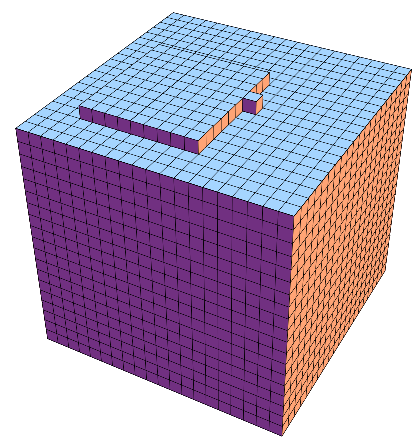
$$\ell_c=\left\lceil\frac{2J}{h}\right\rceil, \qquad m_c=\left\lceil\frac{4J}{h}\right\rceil,$$

are the two-dimensional critical droplet size on a face, respectively, the three-dimensional critical droplet size. Note that $m_c\in\{2\ell_c-1,2\ell_c\}$.

(b) Let $\mathcal{Q}^{1\mathrm{pr}}$ be the set of configurations obtained from $\mathcal{Q}$ by adding a single protuberance anywhere to one of the longest sides of the quasi-square.

(c) Let

$$\begin{aligned}\Gamma^\star = \Gamma^\star(\boxminus,\boxplus) &= H\left(\mathcal{Q}^{\mathrm{1pr}}\right) - H(\boxminus)\\ &= J[2m_c(m_c-\delta_c) + 2m_c(m_c-1) \\ &\qquad + 2(m_c-\delta_c)(m_c-1) + 4\ell_c] \\ &\quad - h[m_c(m_c-\delta_c)(m_c-1) + \ell_c(\ell_c-1)+1].\end{aligned}$$



An element of $\mathcal{Q}^{\mathrm{1pr}}$ for $\ell_c = 10$, $m_c = 20$ and $\delta_c = 0$.

THEOREM 7.2 carries over: the proof of (H1–H2) is the same.

THEOREM 7.3 carries over: $\mathcal{P}^\star(\boxminus,\boxplus) = \mathcal{Q}$ and $\mathcal{C}^\star(\boxminus,\boxplus) = \mathcal{Q}^{\text{1pr}}$.

THEOREM 7.4: the prefactor K can again be computed explicitly, namely,

$$K = K_3 = \frac{K_2}{M_3}$$

with K_2 the prefactor in two dimensions and M_3 the total number of quasi-cubes in three dimensions contained in a three-dimensional critical droplet.

The rationale is that a three-dimensional critical droplet is obtained by first growing a quasi-cube with the appropriate side lengths and then growing a two-dimensional critical droplet on an appropriate side of this quasi-cube.

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(4) G. Ben Arous, R. Cerf, Metastability of the three-dimensional Ising model on a torus at very low temperatures, Electron. J. Probab. 1 (1996), article 10, 1–55.

LITERATURE:

Chapter 17 in Bovier and den Hollander 2015, and references therein.