# LECTURE 15

## Configuration random graphs

## § TARGET

In this lecture we focus on the metastable behaviour at low temperature of Glauber spin-flip dynamics on sparse random graphs. We pick a large number of vertices and randomly allocate edges according to the Configuration Model with a prescribed degree distribution.

S. Dommers, F. den Hollander, O. Jovanovski, F.R. Nardi 2017

In this setting we have the universal theorems available that were described and proved in Lectures 5–6. All we need to do is verify the hypotheses that were needed there, and identify the key quantities in the theorems:

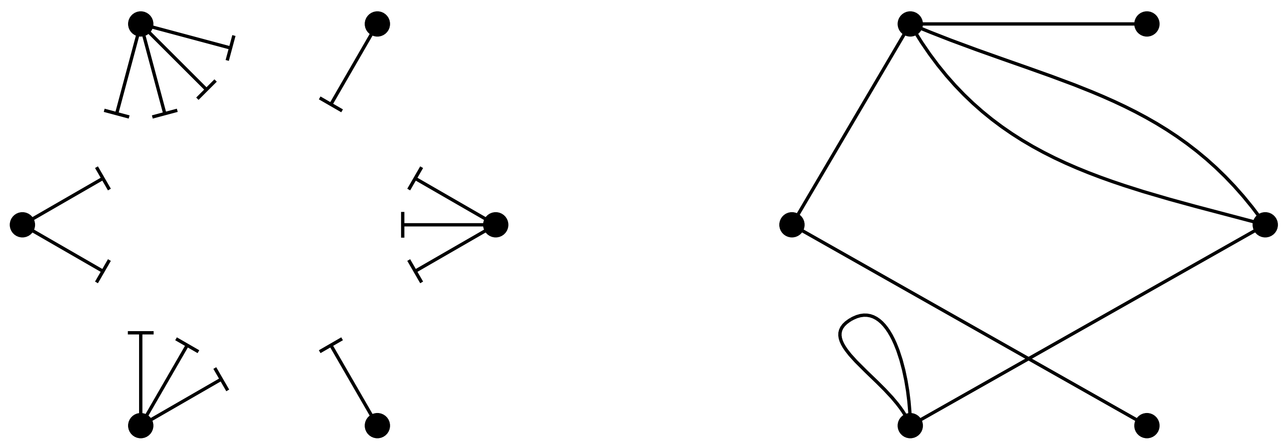
- communication height
- critical droplet
- prefactor

We will see that these are non-trivial objects. The critical droplets, representing the saddle points for the crossover, have a size that is of the order of the number of vertices. The reason is that the random graphs generated by the Configuration Model are expander graphs.

As we already saw in Lectures 13–14, the random graphs represent a random environment for the dynamics, because the pair interactions in the Hamiltonian only act along the edges that are present.

What distinguishes CM from ER and CL is that the graphs are sparse rather than dense.

## ALGORITHM FOR CONFIGURATION RANDOM GRAPH:



size 6

degrees $(1, 3, 1, 3, 2, 4)$

randomly pair half-edges

§ MODEL

1. Given a finite connected non-oriented multigraph

$$G = (V, E),$$

the Hamiltonian is

$$H(\xi) = -\frac{J}{2} \sum_{(v,w)\in E} \xi(v)\xi(w) - \frac{h}{2} \sum_{v \in V} \xi(v), \qquad \xi \in \Omega,$$

with $J > 0$ the ferromagnetic pair potential and $h > 0$ the magnetic field.

The first sum in the right-hand side runs over all non-oriented edges in $E$. Hence, if $v, w \in V$ have $k \in \mathbb{N}_0$ edges between them, then their joint contribution to the energy is $-k\frac{J}{2}\xi(v)\xi(w)$.

2. We write $\xi \sim \xi'$ if and only if $\xi$ and $\xi'$ agree at all but one vertex. A transition from $\xi$ to $\xi'$ corresponds to a flip of a single spin, and is referred to as an allowed move.

We write $\mathbb{P}_\xi^{G,\beta}$ to denote the law of $(\xi_t)_{t\geq 0}$ given $\xi_0 = \xi$, and $\lambda^{G,\beta}$ to denote the principal eigenvalue of the generator of the dynamics.

The upper indices $G, \beta$ exhibit the dependence on the underlying graph $G$ and the interaction strength $\beta$ between neighbouring spins.

3. Recall the geometric definitions in Lecture 5.

It is easy to check that $S_{\text{stab}} = \{\boxplus\}$ for all $G$ because $J, h > 0$. For general $G$, however, $S_{\text{meta}}$ is not a singleton, but we will be interested in those $G$ for which the following hypothesis is satisfied:

$$\text{(H1)} \qquad S_{\text{meta}} = \{\boxminus\}.$$

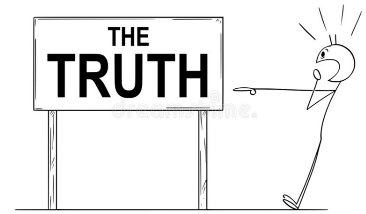
The energy barrier between $\boxminus$ and $\boxplus$ is

$$\Gamma^\star = \Phi(\boxminus, \boxplus) - H(\boxminus).$$

Recall also the definition of $(\mathcal{P}^\star, \mathcal{C}^\star)$, the protocritial and critical set.

As shown in Lectures 5–6, subject to hypothesis (H1) the following three theorems hold.

## § UNIVERSAL THEOREMS




### THEOREM 15.1

$\lim_{\beta\to\infty} \mathbb{P}^{G,\beta}_{\boxminus}(\tau_{\mathcal{C}^\star} < \tau_{\boxplus} \mid \tau_{\boxplus} < \tau_{\boxminus}) = 1.$

### THEOREM 15.2

*There exists a* $K^\star \in (0,\infty)$ *such that*

$$\lim_{\beta\to\infty} \mathrm{e}^{-\beta\Gamma^\star}\, \mathbb{E}^{G,\beta}_{\boxminus}(\tau_{\boxplus}) = K^\star.$$

### THEOREM 15.3

(a) $\lim_{\beta\to\infty} \lambda^{G,\beta}\, \mathbb{E}^{G,\beta}_{\boxminus}(\tau_{\boxplus}) = 1.$

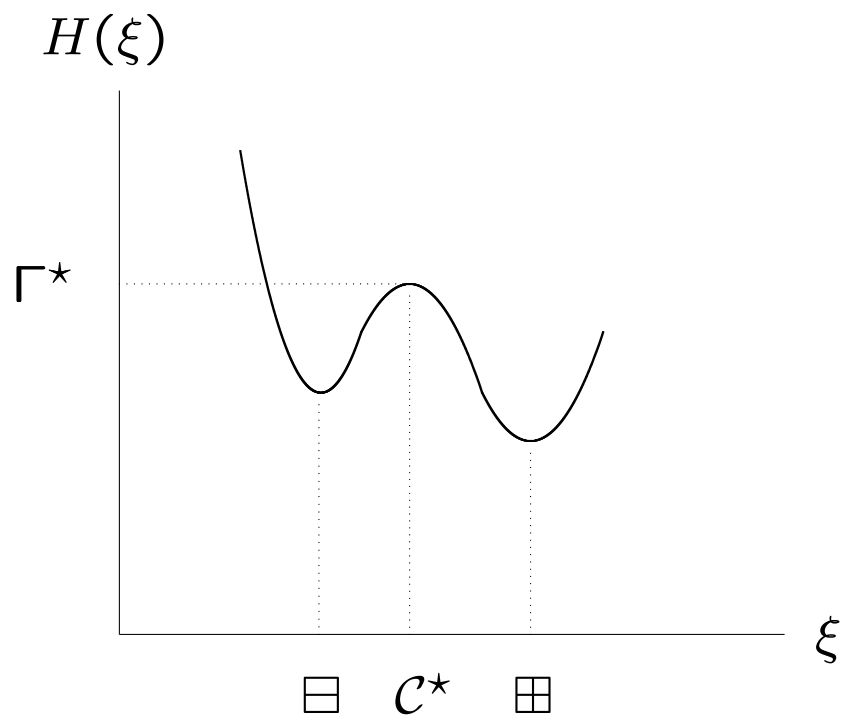
(b) $\lim_{\beta\to\infty} \mathbb{P}^{G,\beta}_{\boxminus}(\tau_{\boxplus}/\mathbb{E}^{G,\beta}_{\boxminus}(\tau_{\boxplus}) > t) = \mathrm{e}^{-t}$ *for all* $t \geq 0$.

The validity of Theorems 15.1–15.3 does not rely on the details of the graph $G$, provided it is finite, connected and non-oriented. For concrete choices of $G$, the task is to verify Hypothesis (H1) and to identify the triple

$$(\mathcal{C}^\star, \Gamma^\star, K^\star).$$

For deterministic graphs this task has been carried out successfully, for a large number of examples. For random graphs, however, the triplet is random, and describing this triplet represents a very serious challenge.

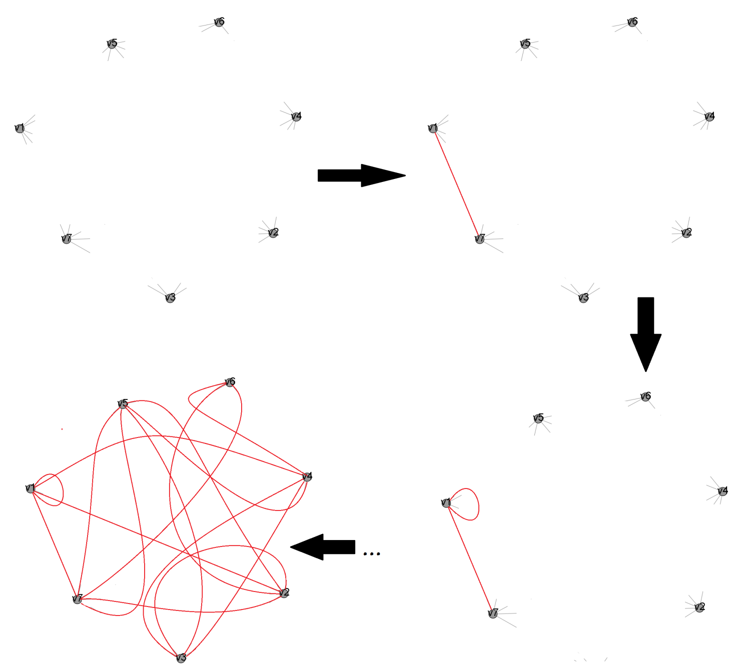
In what follows we focus on the particular class of random graphs called the Configuration Model.




Schematic picture of $H$ and $\boxminus, \boxplus$ and $\Gamma^\star, \mathcal{C}^\star$.

## § CONFIGURATION MODEL

We recall the construction of the random multi-graph known as the Configuration Model.



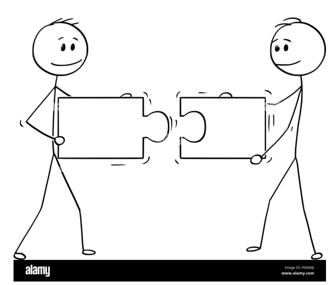
Three steps in the pairing of half-edges for $N = 7$ and degree sequence $(5, 5, 4, 5, 5, 3, 5)$.

1. Fix $N \in \mathbb{N}$. With each vertex $i \in [N]$ we associate a random degree $d_i$, in such a way that

$$(d_i)_{i \in [N]}$$

are i.i.d. with probability distribution $f$ conditional on the event $\{\sum_{i \in [N]} d_i = \text{even}\}$. Consider a uniform matching of the half-edges, leading a multi-graph $\text{CM}_N$ satisfying the requirement that the degree of vertex $i$ is $d_i$ for $i \in [N]$.

The total number of edges is $\frac{1}{2} \sum_{i \in [N]} d_i$.



2. Throughout the sequel we use the symbol $\mathbb{P}_N$ to denote the law of the random multi-graph $\text{CM}_N$ generated by the Configuration Model.

3. To avoid degeneracies we assume that

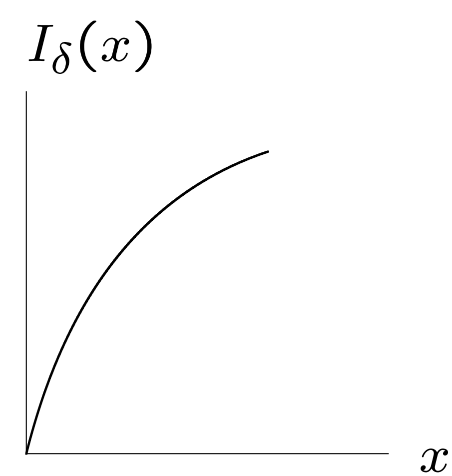
$$\begin{aligned} d_{\text{min}} &= \min\{k \in \mathbb{N}\colon\ f(k) > 0\} \geq 3, \\ d_{\text{ave}} &= \sum_{k \in \mathbb{N}} k f(k) < \infty, \end{aligned}$$

i.e., all degrees are at least three and the average degree is finite. In this case $\text{CM}_N$ is connected with high probability (whp), i.e., with probability tending to 1 as $N \to \infty$.

4. Along the way we need a technical function that allows us to quantify certain properties of the energy landscape, which we introduce next. Later we provide the underlying heuristics.

For $x \in (0, \frac{1}{2}]$ and $\delta \in (1, \infty)$, define

$$I_\delta(x) = \inf \Big\{ y \in (0, x] \colon 1 < x^{x(1-1/\delta)} (1-x)^{(1-x)(1-1/\delta)} \times (1-x-y)^{-(1-x-y)/2} (x-y)^{-(x-y)/2} y^{-y} \Big\}.$$




Plot of the function $x \mapsto I_\delta(x)$ for $\delta = 6$.

## § MAIN THEOREMS

We want to prove Hypothesis (H1) and also to identify the critical triplet for $\mathrm{CM}_N$, which we henceforth denote by $(\mathcal{C}_N^\star, \Gamma_N^\star, K_N^\star)$, in the limit as $N \to \infty$.

Our first theorem settles Hypothesis (H1) for small $h/J$. Suppose that

$$\frac{h}{J} < \frac{2I_{d_{\text{ave}}}\left(\frac{1}{2}\right) - \frac{1}{2}\left(1 - 4I_{d_{\min}}\left(\frac{1}{2}\right)\right)^2\left(1 - 2I_{d_{\min}}\left(\frac{1}{2}\right)\right)^{-1}}{\left(\frac{1}{d_{\text{ave}}} + \frac{1}{2}\right)}.$$

THEOREM 15.4

*If the above inequality is satisfied, then*

$$\lim_{N\to\infty} \mathbb{P}_N\Big(\mathrm{CM}_N \text{ satisfies (H1)}\Big) = 1.$$

Our second and third theorem provide upper and lower bounds on $\Gamma_N^\star$. Label the vertices of the graph in order of increasing degree. Let $\gamma\colon \boxminus \to \boxplus$ be the path that successively flips the vertices $1, \ldots, N$ (in that order), and for $M \in [N]$ let $\ell_M = \sum_{i \in [M]} d_i$.

THEOREM 15.5

*Define*

$$\bar{M} = \bar{M}(h/J)$$
$$= \min \left\{ M \in [N] \colon \frac{h}{J} \geq \ell_{M+1} \left(1 - \frac{\ell_{M+1}}{\ell_N}\right) - \ell_M \left(1 - \frac{\ell_M}{\ell_N}\right) \right\},$$

*and note that $\bar{M} < N/2$. Then whp*

$$\Gamma_N^\star \leq \Gamma_N^+, \qquad \Gamma_N^+ = J \ell_{\bar{M}} \Big(1 - \frac{\ell_{\bar{M}}}{\ell_N}\Big) - h\bar{M} \pm O\Big(\ell_N^{3/4}\Big).$$

THEOREM 15.6

*Define*

$$\tilde{M} = \min \left\{ M \in [N] \colon \ \ell_M \geq \tfrac{1}{2}\ell_N \right\}.$$

*Then whp*

$$\Gamma_N^\star \geq \Gamma_N^-, \qquad \Gamma_N^- = J\, d_{\mathrm{ave}}\, I_{d_{\mathrm{ave}}}\left(\tfrac{1}{2}\right) N - h\tilde{M} - o(N).$$

COROLLARY 15.7

*Under Hypothesis* (H), *Theorems 15.5–15.6 yield*

$$\lim_{\beta\to\infty} \mathbb{P}_{\boxminus}^{G,\beta}\left( e^{\Gamma_N^- - \varepsilon} \leq \tau_{\boxplus} \leq e^{\Gamma_N^+ + \varepsilon} \right) = 1 \quad \forall\, \varepsilon > 0$$

REMARK: For simple degree distributions, like Dirac or power law, the quantities $\bar{M}$, $\ell_{\bar{M}}$, $\tilde{M}$ can be computed explicitly.

The bounds in Theorems 15.5–15.6 are tight in the limit of large degrees. Indeed, by the law of large numbers we have that

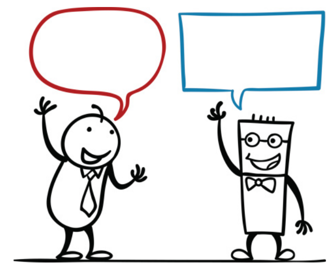
$$\ell_N \frac{\ell_{\bar{M}}}{\ell_N}\left(1 - \frac{\ell_{\bar{M}}}{\ell_N}\right) \leq \tfrac{1}{4}\ell_N = \tfrac{1}{4} d_{\text{ave}}\, N \left[1 + o(1)\right].$$

Hence

$$\frac{\Gamma_N^+}{\Gamma_N^-} = \frac{\tfrac{1}{4} d_{\text{ave}} \left[1 + o\,(1)\right] - \tfrac{h}{J}\tfrac{\bar{M}}{N} + o(1)}{d_{\text{ave}} I_{d_{\text{ave}}}\left(\tfrac{1}{2}\right) - \tfrac{h}{J}\tfrac{\tilde{M}}{N} - o(1)}.$$

In the limit as $d_{\text{ave}} \to \infty$ we have $I_{d_{\text{ave}}}\left(\tfrac{1}{2}\right) \to \tfrac{1}{4}$, in which case the above ratio tends to 1.

## § DISCUSSION



1. The integer $\bar{M}$ has the following interpretation. The path $\gamma : \boxminus \to \boxplus$ is obtained by flipping $(-1)$-valued vertices to $(+1)$-valued vertices in order of increasing degree. Up to fluctuations of size $o(N)$, the energy along $\gamma$ increases for the first $\bar{M}$ steps and decreases for the remaining $N - \bar{M}$ steps.

2. The integer $\tilde{M}$ has the following interpretation. To obtain our lower bound on $\Gamma^{\star}_{N}$ we consider configurations whose $(+1)$-valued vertices have total degree at most $\frac{1}{2}\ell_N$. The total number of $(+1)$-valued vertices in such type of configurations is at most $\tilde{M}$.

3. If we consider all sets on $\mathrm{CM}_N$ that are of total degree $x\ell_N$ and share $y\ell_N$ edges with their complement, then $I_\delta(x)$ represents (a lower bound on) the least value for $y$ such that the average number of such sets is at least 1. In particular, for smaller values of $y$ this average number is exponentially small.

4. We believe that Hypothesis (H1) holds as soon as

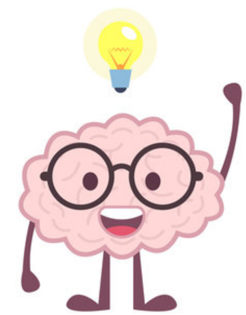
$$0 < h < (d_{\mathrm{min}} - 1)J,$$

i.e., we believe that in the limit as $\beta \to \infty$ followed by $N \to \infty$ this choice of parameters corresponds to the metastable regime of our dynamics, i.e., the regime where $(\boxminus, \boxplus)$ is a metastable pair.

5. The scaling behaviour of $\Gamma_N^\star, K_N^\star$ as $N \to \infty$, as well as the geometry of $\mathcal{C}_N^\star$, are hard to capture.

## CONJECTURE 15.8

*There exists a* $\gamma^\star \in (0,\infty)$ *such that*



$$\lim_{N\to\infty} \mathbb{P}_N\Big(\Big|N^{-1}\Gamma_N^\star - \gamma^\star\Big| > \delta\Big) = 0 \qquad \forall\,\delta > 0.$$

## CONJECTURE 15.9

*There exists a* $c^\star \in (0,1)$ *such that*

$$\lim_{N\to\infty} \mathbb{P}_N\Big(\Big|N^{-1}\log|\mathcal{C}_N^\star| - c^\star\Big| > \delta\Big) = 0 \qquad \forall\,\delta > 0.$$

## CONJECTURE 15.10

*There exists a* $\kappa^\star \in (1,\infty)$ *such that*

$$\lim_{N\to\infty} \mathbb{P}_N\Big(\Big||\mathcal{C}_N^\star|\,K_N^\star - \kappa^\star\Big| > \delta\Big) = 0 \qquad \forall\,\delta > 0.$$

6. It is shown in Dommers 2017 that for a random regular graph with degree $r \geq 3$, there exist constants $0 < \gamma_-^\star(r) < \gamma_+^\star(r) < \infty$ such that

$$\lim_{N\to\infty} \lim_{\beta\to\infty} \mathbb{E}_N \left( \mathbb{P}_{\boxminus}^{\mathsf{CM}_N} \left( \mathrm{e}^{\beta N \gamma_-^\star(r)} \leq \tau_{\boxplus} \leq \mathrm{e}^{\beta N \gamma_+^\star(r)} \right) \right) = 1$$

when $\frac{h}{J} \in (0, C_0\sqrt{r})$ for some constant $C_0 \in (0,\infty)$ that is small enough.

Moreover, there exist constants $C_1 \in (0, \frac{1}{4}\sqrt{3})$ and $C_2 \in (0,\infty)$ (depending on $C_0$) such that

$$\gamma_-^\star(r) \geq \tfrac{1}{4}Jr - C_1 J\sqrt{r}, \qquad \gamma_+^\star(r) \leq \tfrac{1}{4}Jr + C_2 J\sqrt{r}.$$

These results are derived without Hypothesis (H1), but it is shown that Hypothesis (H1) holds as soon as $r \geq 6$.

PAPERS:

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(3) S. Dommers, F. den Hollander, O. Jovanovski, F.R. Nardi,
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