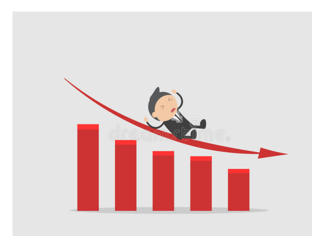
LECTURE 11

Widom-Rowlinson model for disks in the plane:
Microscopic fluctuations of the critical droplet

§ GOAL

In Lecture 10 we analysed the mesoscopic fluctuations of the surface of the critical droplet. This was achieved with the help of certain large deviation principles and moderate deviation principles for the shape and the volume of the halo of the configuration in the Widom-Rowlinson model (WRM).

The theorems formulated in Lecture 10 required three technical lemmas, whose proof in turn relies on a closer analysis of the microscopic fluctuations of the surface of the critical droplet. The goal of this lecture is to describe these fluctuations in some detail and explain what role they play.



§ PARABOLIC INTERFACE MODEL

In this lecture we introduce a certain Gibbs modification of what in stochastic geometry is called the paraboloid hull process.

This modification, which we refer to as the parabolic interface model (PIM), arises as the scaling limit of the surface of the critical droplet in the Widom-Rowlinson model (WRM), in the limit of low temperature.

In what follows we first focus on the PIM, which is of special interest because it provides a rigorous microscopic foundation for the notion of surface tension in statistical physics. Towards the end of the lecture we return to the WRM and indicate how the results for the PIM can be used to describe the microscopic fluctuations of the surface of the critical droplet in the WRM.

DEFINITION 11.1 Extremality

Let $\mathbf{x} = \{(s_i, y_i)\}_{i \in I}$ be a sequence of points in $\mathbb{R}^2$ indexed by a countable set of successive indices $I \subseteq \mathbb{Z}$. Consider the union of downward parabolas with tips at $\mathbf{x}$:

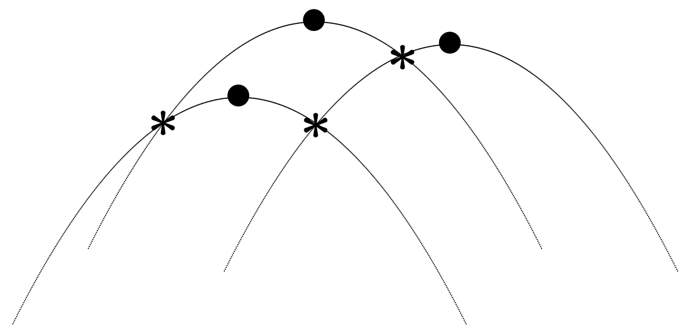
$$\mathcal{A}(\mathbf{x}) = \bigcup_{i \in I} \Big\{(s, y) \in \mathbb{R}^2 \colon\ y \leq y_i - \tfrac{1}{2}(s - s_i)^2\Big\}.$$

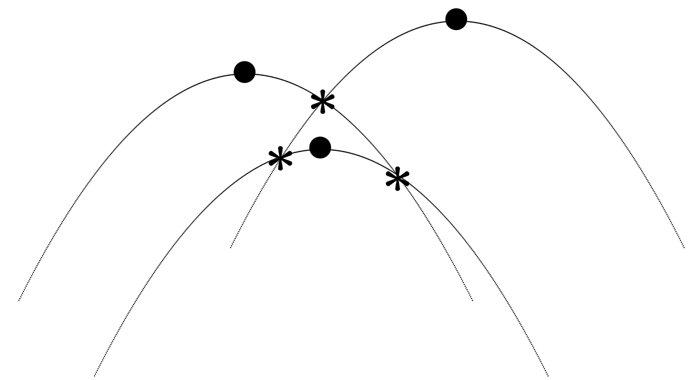
(i) $(s_i, y_i) \in \mathbf{x}$ is said to be extremal when

$$\mathcal{A}(\mathbf{x} \setminus \{(s_i, y_i)\}) \subsetneq \mathcal{A}(\mathbf{x}).$$

(ii) $\mathbf{x} = \{(s_i, y_i)\}_{i \in I}$ is said to be extremal if every $(s_i, y_i) \in \mathbf{x}$ is extremal.

We use $\mathcal{E}$ to denote the set of all extremal $\mathbf{x}$. These will be referred to as sequences of boundary points.





Two examples of triplets of parabolas. The $\bullet$'s mark the tips of the parabolas, the $*$'s mark the pairwise intersection points.

The left picture is extremal, and the two outer $*$'s appear in the same order as the associated pairs of $\bullet$'s. The right picture is not extremal, and the two outer $*$'s appear in the opposite order as the associated pairs of $\bullet$'s.

In what follows we will consider three Gibbs measures and associated partition functions for the PIM that are linked to each other and are all needed to identify the free energy of the PIM:

- grand-canonical pinned partition function
- canonical unpinned partition function
- canonical pinned partition function

The first is the object we are after, while the second and third are needed to identify the free energy associated with the first, which will capture the surface free energy of the critical droplet in the WRM.

§ KEY PARTITION FUNCTION AND GIBBS MEASURE

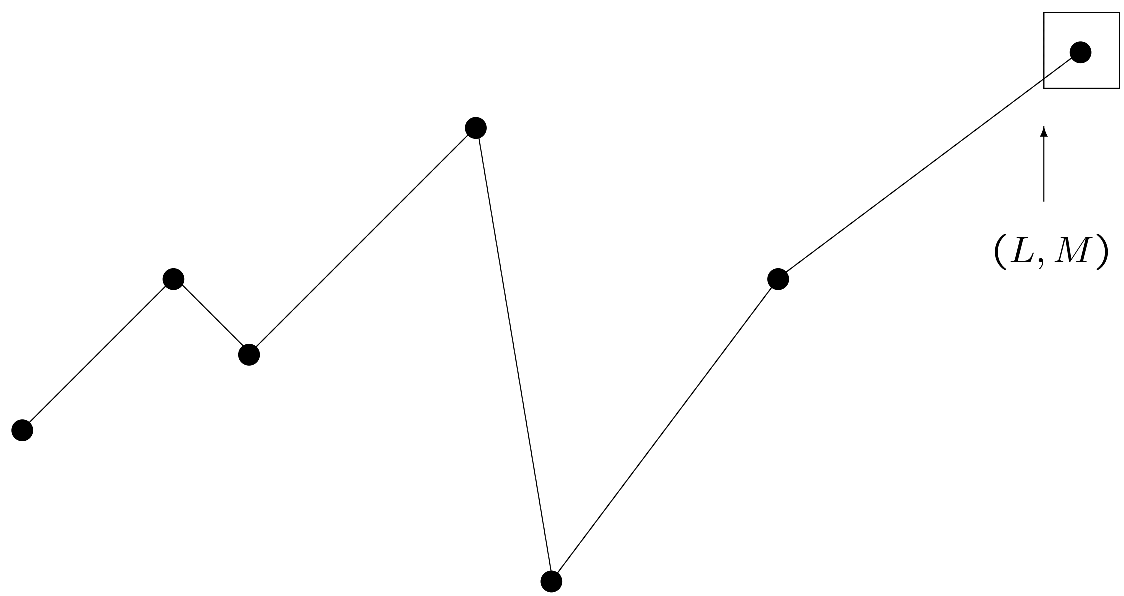
The following partition function associated with the PIM, called the grand-canonical pinned partition function, will be the key instrument in our analysis of the WRM. For $L \in \mathbb{N}_0$ and $M \in \mathbb{Z}$, define

$$\mathcal{Z}_{L,M} = 1 + \sum_{n=2}^{\infty} \frac{L}{n} \int_{\mathbb{R}_+^{n-1}} \mathrm{d}s_2 \cdots \mathrm{d}s_n \int_{\mathbb{R}^{n-1}} \mathrm{d}y_2 \cdots \mathrm{d}y_n$$
$$\times 1_{\{s_1 < s_2 < \cdots < s_n < s_{n+1}\}} 1_{\mathcal{E}}\Big(\{(s_i, y_i)\}_{i=1}^{n+1}\Big)$$
$$\times \exp\Big(-\sum_{i=1}^{n} \frac{(s_{i+1}-s_i)^3}{24} - \sum_{i=1}^{n} \frac{(y_{i+1}-y_i)^2}{2(s_{i+1}-s_i)}\Big)$$

with the convention

$$(s_1, y_1) = (0,0), \quad (s_{n+1}, y_{n+1}) \in \mathcal{T}_{L,M}.$$

with $\mathcal{T}_{L,M}$ the unit square with lower-left corner at (L, M).



A trajectory of tips of the downward parabolas. The number of tips is variable. The last tip ends in $\mathcal{T}_{L,M}$.

The indicator $1_{\mathcal{E}}$ ensures the extremality of all the points (s_i, y_i), $i = 1, \ldots, n+1$, with respect to the strip

$$H_{(0,L)} = \{(s, y)\colon\ s \in (0, L), y \in \mathbb{R}\},$$

i.e., the condition in Definition 11.1(i) is replaced by

$$\mathcal{A}(\mathbf{x} \setminus \{(s_i, y_i)\}) \cap H_{(0,L)} \subsetneq \mathcal{A}(\mathbf{x}) \cap H_{(0,L)}.$$

In other words, we cut out the parts of the parabolas that stick out of the strip.

Later we will replace the open boundary condition by a circular boundary condition, appropriate for the circular interface of the critical droplet in the WRM.

The exponential factor represents a Gibbs modification of the standard interface model as it arises from the WRM via an expansion of the halo volume sticking out of the critical droplet, as mentioned in Lecture 10.

Later we wil make an appropriate choice for L, M to link PIM and WRM, in particular

$$L = 2\pi G_\kappa \beta^{1/3}, \qquad M = 0,$$

with $G_\kappa = (2\kappa)^{2/3}/(\kappa - 1)$.

Think of the trajectory as what we see when we zoom in on the boundary points of the critical droplet until the scale of their separation is order 1. On that scale the interface looks flat.

REMARK:

The extremality constraint $\mathcal{E}$ in the definition of the PIM can be shown to be a local constraint, namely, the location of any extremal point is constrained by the location of its two neighbours only. This is a crucial property because it allows us to use transfer operators, as we will see later.

§ COSMETICS

Some cosmetics is needed before we go on. It is expedient to move away from labelled tips, and instead work with horizontal displacements and equivalence classes of vertical heights, as explained in 1 and 2 below.

1. Let $\mathscr{X}$ be the set of $\mathbf{x} = \{(s_i, y_i)\}_{i \in I}$ such that for every $s \in \mathbb{R}$ there is at most one point $(s_i, y_i) \in \mathbf{x}$ with $s = s_i$. The set $\mathscr{X}$ is equipped with the σ-algebra $\mathscr{G}$ generated by the counting variables $B \mapsto |\mathbf{x} \cap B|$ with $B \subset \mathbb{R}^2$ Borel.

The Gibbs measure $\mathcal{P}_{L,M}$ associated with the partition function $\mathcal{Z}_{L,M}$ is the probability measure on $(\mathcal{X}, \mathscr{G})$ defined for $G \in \mathscr{G}$ by

$$
\begin{aligned}
&\mathcal{P}_{L,M}(G) \\
&= \frac{1}{\mathcal{Z}_{L,M}} \Bigg(1_G(\varnothing) + \sum_{n=2}^{\infty} \frac{L}{n} \int_{\mathbb{R}_+^{n-1}} \mathrm{d}s_2 \cdots \mathrm{d}s_n \int_{\mathbb{R}^{n-1}} \mathrm{d}y_2 \cdots \mathrm{d}y_n \\
&\times 1_{\{s_1 < s_2 < \cdots < s_n < s_{n+1}\}} 1_{\mathcal{E} \cap G}\Big(\{(s_i, y_i)\}_{i=1}^{n+1} \Big) \Big) \\
&\times \exp\Big(- \sum_{i=1}^{n} \frac{(s_{i+1} - s_i)^3}{24} - \sum_{i=1}^{n} \frac{(y_{i+1} - y_i)^2}{2(s_{i+1} - s_i)} \Big) \Bigg)
\end{aligned}
$$

and represents the law of the tips of the parabolas.

2. Identify configurations that can be obtained from one another by a global shift of the interface height. To that end, consider the class of $\mathscr{G}$-measurable functions $F\colon \mathcal{X} \to \mathbb{R}$ satisfying

$$(*) \qquad \begin{array}{l} F\Big(\{(s_i, m + y_i)\}_{i\in I}\Big) = F\Big(\{(s_i, y_i)\}_{i\in I}\Big), \\ m \in \mathbb{R},\ \{(s_i, y_i)\}_{i\in I} \in \mathcal{X}. \end{array}$$

Define

$$\mathscr{H} = \sigma\Big(F\colon \mathcal{X} \to \mathbb{R}\colon\ F \text{ is } \mathscr{G}\text{-measurable and satisfies } (*)\Big)$$

and note that $\mathscr{H} \subset \mathscr{G}$. Restricting $\mathcal{P}_{L,M}$ to $\mathscr{H}$, we may view it as a probability measure on $(\mathcal{X}, \mathscr{H})$.

§ TWO KEY THEOREMS

THEOREM 11.2 Interface free energy

For $p, q \in \mathbb{R}$*, let* $\lambda(p,q)$ *be the principal eigenvalue of the operator* $\mathbf{K}_{p,q}$ *defined below. Then, for any* $\alpha \in (-\pi, \pi)$*,*

$$\lim_{L\to\infty} \frac{1}{L} \log \mathcal{Z}_{L, L\tan\alpha} = -p_\alpha - q_\alpha \tan\alpha,$$

where $(p_\alpha, q_\alpha) \in \mathbb{R}\times\mathbb{R}$ *is the unique solution of the set of equations*

$$\log\lambda(p,q) = 0, \qquad \frac{\frac{\partial}{\partial q}\log\lambda(p,q)}{\frac{\partial}{\partial p}\log\lambda(p,q)} = \tan\alpha.$$

If $\alpha = 0$ *(flat interface), then* $(p_\alpha, q_\alpha) = (p_*, 0)$ *with* $p_* \in \mathbb{R}$ *the unique solution of the equation* $\lambda(p, 0) = 1$*.*

Given $\mathbf{x} = \{(s_i, y_i)\}_{i=1}^{n+1}$, define a path $X_L \colon [0, 2\pi] \to \mathbb{R}$ by putting

$$X_L(0) = 0, \quad X_L\Big(2\pi \frac{s_i}{L}\Big) = \frac{y_i}{\sqrt{L}}, \quad 2 \le i \le n, \quad X_L(2\pi) = 0,$$

and applying piecewise affine interpolation.

THEOREM 11.3 Invariance principle

As $L \to \infty$, $X_L = (X_L(t))_{t \in [0,2\pi]}$ under the Gibbs measure $\mathcal{P}_{L,0}$ converges in distribution to σB for some $\sigma^2 \in (0, \infty)$, where $B = (B_t)_{t \in [0,2\pi]}$ is the mean-centered Brownian bridge that was introduced in Lecture 10.

Theorems 11.2–11.3 are crucial ingredients for the proof of the three technical conditions (C1)–(C3) used in Lecture 10.

§ TWO AUXILIARY PARTITION FUNCTIONS

▶ Canonical unpinned partition function

For fixed n, we drop the constraint $(s_{n+1}, y_{n+1}) \in \mathcal{T}_{L,M}$, but use the parameters $p, q \in \mathbb{R}$ to weigh the endpoint (s_{n+1}, y_{n+1}) instead.

This represents a free boundary condition in which we retain only the initial condition $(s_1, y_1) = (0, 0)$.

The resulting partition sum is

$$\mathscr{Z}_n(p,q) = \int_{\mathbb{R}_+^n} \mathrm{d}s_2 \dots \mathrm{d}s_{n+1} \int_{\mathbb{R}^n} \mathrm{d}y_2 \dots \mathrm{d}y_{n+1}$$

$$1_{\{0<s_2<\cdots<s_{n+1}\}} 1_{\mathcal{E}}\Big(\{(s_i,y_i)\}_{i=1}^{n+1}\Big)$$

$$\exp\Big(p s_{n+1} + q y_{n+1} - \sum_{i=1}^{n} \frac{(s_{i+1}-s_i)^3}{24} - \sum_{i=1}^{n} \frac{(y_{i+1}-y_i)^2}{2(s_{i+1}-s_i)}\Big)$$

$$= \int_{\mathbb{R}_+^n} \mathrm{d}\vartheta_1 \dots \mathrm{d}\vartheta_n \int_{\mathbb{R}^n} d\varphi_1 \dots \mathrm{d}\varphi_n$$

$$1_{\mathcal{E}}\Big(\{(\vartheta_i,\varphi_i)\}_{i=1}^{n}\Big)$$

$$\exp\Big(p \sum_{i=1}^{n} \vartheta_i + q \sum_{i=1}^{n} \varphi_i - \sum_{i=1}^{n} \frac{\vartheta_i^3}{24} - \sum_{i=1}^{n} \frac{\varphi_i^2}{2\vartheta_i}\Big).$$

Write $\mathscr{P}_n(p,q)$ to denote the associated n-particle Gibbs measure. Note that $(p,q) \mapsto \log \mathscr{Z}_n(p,q)$ is convex because linear combinations of log-convex functions are log-convex.

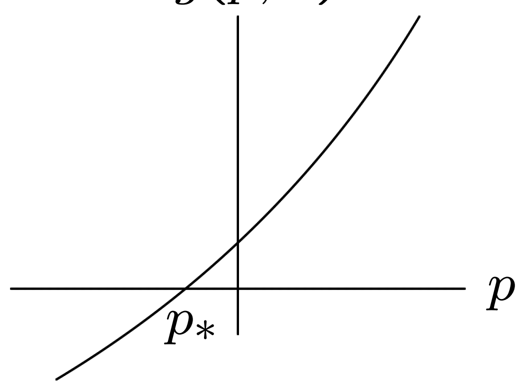
LEMMA 11.4 Interface Gibbs free energy

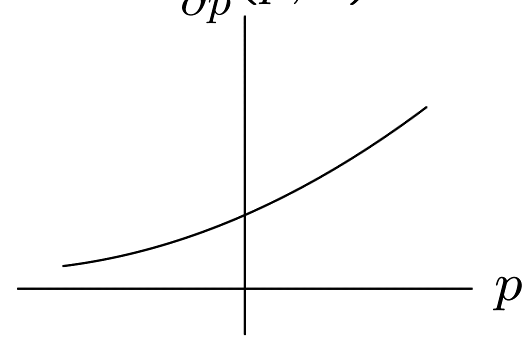
(1) *For every* $(p,q) \in \mathbb{R} \times \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathscr{Z}_n(p,q) = g(p,q),$$

where $g(p,q) = \log \lambda(p,q)$ *is the logarithm of the principal eigenvalue of* $\mathbf{K}_{p,q}$.

(2) $(p,q) \mapsto g(p,q)$ *is analytic, strictly increasing and strictly convex on* $\mathbb{R} \times \mathbb{R}$.





Cartoon of the graph of $p \mapsto g(p,0)$ and $p \mapsto \frac{\partial g}{\partial p}(p,0)$.

▶ Canonical pinned partition function

Define

$$Z_n^{(L,M)} = \int_{(0,L)^n} \mathrm{d}\vartheta_1 \dots \mathrm{d}\vartheta_n \int_{\mathbb{R}^n} \mathrm{d}\varphi_1 \dots \mathrm{d}\varphi_n$$
$$\times\, 1_{\left\{(\sum_{i=1}^n \vartheta_i, \sum_{i=1}^n \varphi_i) \in \mathcal{T}_{L,M}\right\}}$$
$$\times\, 1_{\mathcal{E}}\Big(\{(\vartheta_i, \varphi_i)\}_{i=1}^n\Big) \exp\Big(-\sum_{i=1}^n \frac{\vartheta_i^3}{24} - \sum_{i=1}^n \frac{\varphi_i^2}{2\vartheta_i}\Big).$$

This differs from the two previous partition functions in that both the number of points and the position of the last point are fixed, and we work with the increments of the coordinates.

Write $P_n^{(L,M)}$ to denote the associated n-particle Gibbs measure.

In the following lemma $\mathcal{L}$ denotes the Legendre-Fenchel transform.

LEMMA 11.5 Interface free energy

(1) *For any* $(\rho, \sigma) \in \mathbb{R}_+ \times \mathbb{R}$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n^{(n\rho, n\sigma)} = -f(\rho, \sigma)$$

with

$$f(\rho, \sigma) = (\mathcal{L}g)(\rho, \sigma) = \sup_{(p,q)\in\mathbb{R}^2} \Big[p\rho + q\sigma - g(p, q)\Big].$$

(2) $(\rho, \sigma) \mapsto f(\rho, \sigma)$ *is analytic and strictly convex on* $\mathbb{R}_+ \times \mathbb{R}$*, and*

$$g(p, q) = (\mathcal{L}f)(p, q) = \sup_{(\rho,\sigma)\in\mathbb{R}_+\times\mathbb{R}} \Big[p\rho + q\sigma - f(\rho, \sigma)\Big].$$

At this point we have identified:

Three free energies associated with the PIM,
for different types of boundary conditions.

The pinned partition functions are defined on a strip. For $M = 0$ they can be put on a circle as well, which is needed for the WRM.

In what follows we will explain:

- What the extremality condition exactly does.
- Identify the operator $\mathbf{K}_{p,q}$.

§ EXTREMALITY

In terms of space and height increments, the extremality constraint reads as follows.

LEMMA 11.6 Extremality for triplets

Let $\mathbf{x} = \{(s_i, y_i)\}_{i \in I} \in \mathcal{X}$. *Assume that* $s_i < s_{i+1}$ *for all* $i \in I$ *and define linear spacings* ϑ_i *and height increments* φ_i *by*

$$\vartheta_i = s_{i+1} - s_i, \quad \varphi_i = y_{i+1} - y_i.$$

Then $\mathbf{x} \in \mathcal{E}$ *if and only if*

$$\frac{2}{\vartheta_i + \vartheta_{i-1}} \left(\frac{\varphi_i}{\vartheta_i} - \frac{\varphi_{i-1}}{\vartheta_{i-1}} \right) < 1 \qquad \forall\, i \in I.$$

PROOF: An exercise with quadratic equations. $\square$

§ TRANSFER OPERATOR

Put $\mathbb{S} = \mathbb{R}_+ \times \mathbb{R}$. For $(p,q) \in \mathbb{R} \times \mathbb{R}$ define the integral kernel $K_{p,q}\colon\ \mathbb{S}^2 \to [0,\infty)$ given by

$$K_{p,q}\Big((\vartheta_1,\varphi_1),(\vartheta_2,\varphi_2)\Big)$$
$$= \exp\Big(\frac{1}{2}\Big[p\vartheta_1 + q\varphi_1 - \frac{\vartheta_1^3}{24} - \frac{\varphi_1^2}{2\vartheta_1}\Big]\Big)\, 1_{\Big\{\frac{2}{\vartheta_1+\vartheta_2}(\frac{\varphi_2}{\vartheta_2} - \frac{\varphi_1}{\vartheta_1})<1\Big\}}$$
$$\times \exp\Big(\frac{1}{2}\Big[p\vartheta_2 + q\varphi_2 - \frac{\vartheta_2^3}{24} - \frac{\varphi_2^2}{2\vartheta_2}\Big]\Big).$$

Let $\mathbf{K}_{p,q}\colon\ L^2(\mathbb{S}) \to L^2(\mathbb{S})$ be the integral operator given by

$$(\mathbf{K}_{p,q}f)(\xi) = \int_{\mathbb{S}} \mathrm{d}\xi'\, K_{p,q}(\xi,\xi')f(\xi').$$

Note that $\mathbf{K}_{p,q}$ is not self-adjoint, because the constraint is not symmetric. The adjoint operator $\mathbf{K}^*_{p,q}$ is defined by

$$(\mathbf{K}^*_{p,q}f)(\xi') = \int_{\mathbb{S}} \mathrm{d}\xi\, f(\xi) K_{p,q}(\xi,\xi').$$

Lemmas 11.7–11.8 below collect the key properties of $K_{p,q}$. For ease of notation we keep p, q fixed and suppress them from the notation.

LEMMA 11.7 Kernel properties

(a) $\int_{\mathbb{S}}\int_{\mathbb{S}} \mathrm{d}\xi\, \mathrm{d}\xi'\, |K(\xi,\xi')|^2 < \infty$.

(b) $K(\xi,\xi') \geq 0$ *for all* $\xi,\xi' \in \mathbb{S}$.

(c) *There exist a measurable set* $S \subset \mathbb{S}$ *with a strictly positive Lebesgue measure and a constant* $\varepsilon > 0$ *such that* $K(\xi,\xi') \geq \varepsilon$ *for all* $\xi,\xi' \in S$.

(d) $K^2(\xi_1,\xi_2) = \int_{\mathbb{S}} \mathrm{d}\xi\, K(\xi_1,\xi)K(\xi,\xi_2) > 0$ *for all* $\xi_1,\xi_2 \in \mathbb{S}$.

Write $\sigma(\mathbf{K})$ for the spectrum of $\mathbf{K}$, and let

$$\lambda = r(\mathbf{K}) = \sup\{|\mu| \colon \mu \in \sigma(\mathbf{K})\} = \lim_{n\to\infty} \|\mathbf{K}^n\|^{1/n}$$

denote the spectral radius of $\mathbf{K}$.

LEMMA 11.8 Spectral properties

(1) $\lambda > 0$ *is an isolated simple eigenvalue in* $\sigma(\mathbf{K})$, *with an eigenvector* $\psi \in L^2(\mathbb{S})$ *that is strictly positive a.e. The same is true for* $\mathbf{K}^*$ *with eigenvector* ψ^*. *It may be assumed that* ψ *and* ψ^* *are normalised as* $(\psi^*, \psi) = 1$.

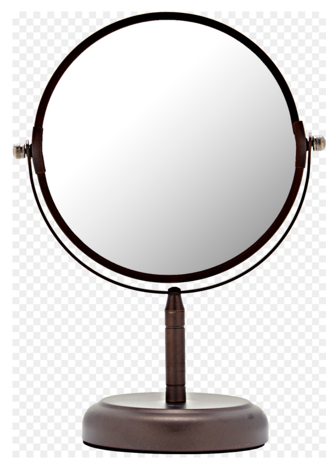
(2) *There exists an* $a \in (0, 1)$ *such that, for all* $f \in L^2(\mathbb{S})$,

$$(f, \mathbf{K}^n f) = \lambda^n \left[(f, \psi)(\psi^*, f) + O(a^n)\|f\|_2^2 \right], \qquad n \to \infty.$$

Lemma 11.8 implies

$$\lim_{n\to\infty} \frac{1}{n} \log(f, \mathbf{K}^n f) = \log \lambda$$

for any $f \in L^2(\mathbb{S})$ with $(f,\psi)(\psi^*, f) \neq 0$. We will see that the scalar product is equal to a partition function for a suitably chosen f encoding the boundary condition.



§ ASSOCIATED MARKOV CHAIN

Define

$$P(\xi, \xi') = \frac{1}{\lambda} \frac{1}{\psi(\xi)} K(\xi, \xi') \, \psi(\xi').$$

Then $P \geq 0$ and

$$\int_{\mathbb{S}} \mathrm{d}\xi' \, P(\xi, \xi') = 1 \qquad \forall \, \xi \in \mathbb{S}.$$

Therefore P is the transition kernel of a Markov chain

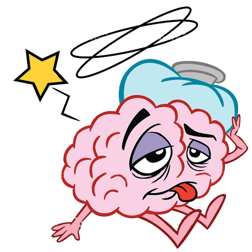
$$X = (X_i)_{i \in \mathbb{N}}$$

with state space $\mathbb{S}$. The stationary distribution π of this Markov chain equals

$$\pi(\xi) = \psi^*(\xi) \psi(\xi), \qquad \xi \in \mathbb{S}.$$

The Markov chain X describe the boundary points of the PIM.

§ PERTURBATION



The WRM is a small perturbation of the circular PIM. To see why we argue as follows, and for simplicity pretend that the dilation parameter m introduced in Lecture 10 is $m = 0$.

1. Let

$$\mathcal{D}_\varepsilon(0) = \{z \in \mathcal{O} \colon\ d_{\mathrm{H}}(\partial S(z), \partial B_{R_c}(0)) \leq \varepsilon\Big\},$$

where d_{H} stands for the Hausdorff distance. It can be shown that, for some $\varepsilon_0 > 0$ and all $0 < \varepsilon \leq \varepsilon_0$,

$$\lim_{\beta\to\infty} \frac{1}{\beta^{1/3}} \log \widehat{\mathbb{P}}\Big(Z^{(0)} \in \mathcal{O},\ Z^{(0)} \notin \mathcal{D}_\varepsilon(0)\Big) = -\infty,$$

where we recall from Lecture 10 that $\widehat{\mathbb{P}}$ is a certain tilted law for the boundary points of the critical droplet in the WRM.

Moreover, if $z = \{z_i\}_{i=1}^n \in \mathcal{D}_\varepsilon(0)$, then as $\varepsilon \downarrow 0$,

$$\max_{1\le i\le n} |\rho_i| = O(\varepsilon), \qquad \max_{1\le i\le n} \theta_i = O(\sqrt{\varepsilon}\,),$$
$$\max_{1\le i\le n} |\rho_{i+1} - \rho_i|/\theta_i = O(\sqrt{\varepsilon}\,), \qquad n^{-1} = O(\sqrt{\varepsilon}),$$

when

$$z_i = (r_i \cos t_i, r_i \sin t_i), \qquad r_i = (R_c - 1) + \rho_i, \qquad \theta_i = t_{i+1} - t_i.$$

Thus, when analysing the probability $\widehat{\mathbb{P}}(Z^{(0)} \in \mathcal{O})$, we may use the a priori bounds.

2. We know that in the circular PIM the constraint $z = \{z_i\}_{i=1}^n \in \mathcal{O}$ is satisfied if and only if all triples (z_{i-1}, z_i, z_{i+1}) are extremal (just as in the PIM). Moreover, we showed that the triple (z_{i-1}, z_i, z_{i+1}) is extremal if and only if

$$t_{i-1,i} < t_{i,i+1},$$

where

$$v_{i-1,i} = \Big(r_{i-1,i}\cos t_{i-1,i}, r_{i-1,i}\sin t_{i-1,i}\Big),$$
$$v_{i,i+1} = \Big(r_{i,i+1}\cos t_{i,i+1}, r_{i,i+1}\sin t_{i,i+1}\Big),$$

are the positions in polar coordinates of the points in $\partial B(z_{i-1}) \cap \partial B(z_i)$, respectively, $\partial B(z_i) \cap \partial B(z_{i+1})$ hat lie in the annulus $A_{R_c,\varepsilon}(0)$. In order to compute $(r_{i-1,i}, t_{i-1,i})$ and $(r_{i,i+1}, t_{i,i+1})$, we use that the a priori bounds imply that, for $1 \le i \le n$,

$$r_{i+1} - r_i = \rho_{i+1} - \rho_i = O(\varepsilon), \quad t_{i+1} - t_i = \theta_i = O(\sqrt{\varepsilon}\,).$$

3. Let $z, z' \in A_{R_c-1,\varepsilon}(0)$ be such that $t < t'$, $r - r' = O(\varepsilon)$ and $t - t' = O(\sqrt{\varepsilon}\,)$. Let $v = (r''\cos t'', r''\sin t'')$ be the position in polar coordinates of the point in $\partial B(z) \cap \partial B(z')$ that lies in $A_{R_c,\varepsilon}(0)$. Then we have the relations

$$1 = (r'' \cos t'' - r \cos t)^2 + (r'' \sin t'' - r \sin t)^2,$$
$$1 = (r'' \cos t'' - r' \cos t')^2 + (r'' \sin t'' - r' \sin t')^2.$$

Writing out the squares, we obtain

$$1 = r''^2 + r^2 - 2r''r \cos(t'' - t),$$
$$1 = r''^2 + r'^2 - 2r''r' \cos(t'' - t').$$

Inserting

$$r = R_c - 1 + \rho, \quad r' = R_c - 1 + \rho', \quad r'' = R_c + \rho'',$$

using that $t < t'' < t'$, $\rho, \rho', \rho'' = O(\varepsilon)$ and $t - t' = O(\sqrt{\varepsilon})$, and expanding the cosine, we get

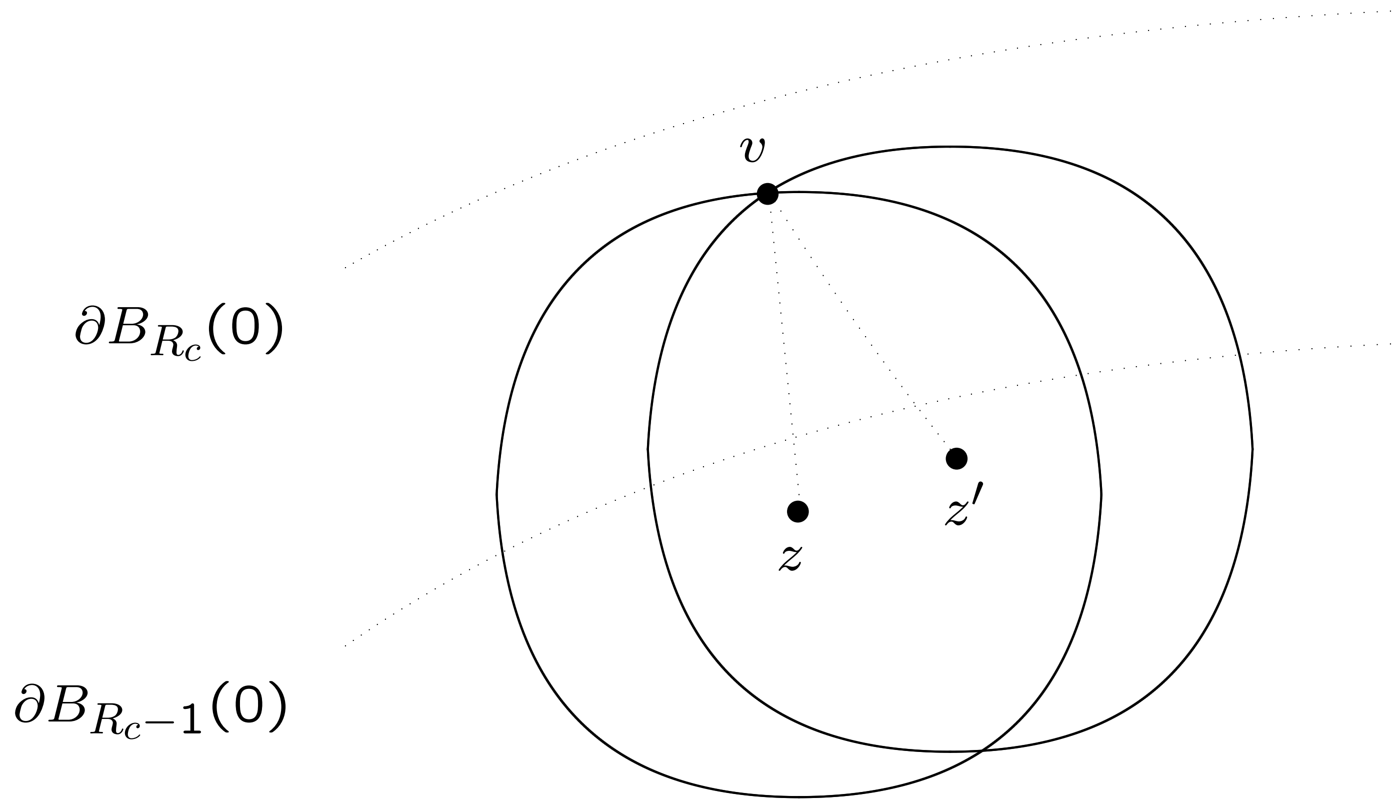
$$\rho'' - \rho = -[1 + O(\varepsilon)]\, R_c(R_c - 1)\,(t'' - t)^2,$$
$$\rho'' - \rho' = -[1 + O(\varepsilon)]\, R_c(R_c - 1)\,(t'' - t')^2.$$

These are two equations for the two unknowns ρ'', t'', which are functions of ρ, t and ρ', t' subject to an error. Since we will have no need for ρ'', we subtract the two equations to get

$$\begin{aligned}\rho' - \rho &= -[1 + O(\varepsilon)]\, R_c(R_c - 1)\, [(t'' - t)^2 - (t'' - t')^2] \\ &= -[1 + O(\varepsilon)]\, R_c(R_c - 1)\, (t' - t)[2t'' - (t' + t)],\end{aligned}$$

or

$$t'' = \frac{t' + t}{2} - [1 + O(\varepsilon)] \frac{1}{2R_c(R_c - 1)} \frac{\rho' - \rho}{t' - t}.$$



Intersecting 1-discs in the boundary layer.

We see that $z = \{z_i\}_{i=1}^{n} \in \mathcal{O}$ if and only if, for $1 \le i \le n$,

$$\frac{t_i + t_{i-1}}{2} - [1 + O(\varepsilon)]\frac{1}{2R_c(R_c - 1)}\frac{\rho_i - \rho_{i-1}}{t_i - t_{i-1}} < \frac{t_{i+1} + t_i}{2} - [1 + O(\varepsilon)]\frac{1}{2R_c(R_c - 1)}\frac{\rho_{i+1} - \rho_i}{t_{i+1} - t_i},$$

which is the same as

$$0 < \frac{t_{i+1} - t_{i-1}}{2} - [1 + O(\varepsilon)]\frac{1}{2R_c(R_c - 1)}\left(\frac{\rho_{i+1} - \rho_i}{t_{i+1} - t_i} - \frac{\rho_i - \rho_{i-1}}{t_i - t_{i-1}}\right).$$

4. Recall from Lecture 10 that the average number of disks touching the boundary of the critical droplet equals $2\pi G_\kappa \beta^{1/3}$ with $G_\kappa = (2\kappa)^{2/3}/(\kappa-1)$. Inserting the scaling

$$\vartheta_i = G_\kappa \beta^{1/3}(t_{i+1} - t_i), \quad \rho_i = \varphi_i/(2\kappa)^{1/3}\beta^{2/3},$$

and using that $1/2R_c(R_c-1) = (\kappa-1)^2/2\kappa$, we see that this becomes

$$0 < \frac{1}{\beta^{1/3}G_\kappa}\left[\frac{\vartheta_i + \vartheta_{i-1}}{2} - [1+O(\varepsilon)]\left(\frac{\varphi_i}{\vartheta_i} - \frac{\varphi_{i-1}}{\vartheta_{i-1}}\right)\right].$$

The prefactor can be dropped, after which we obtain

$$\frac{2}{\vartheta_i + \vartheta_{i-1}}\left(\frac{\varphi_i}{\vartheta_i} - \frac{\varphi_{i-1}}{\vartheta_{i-1}}\right) < 1 + O(\varepsilon).$$

Apart from the error term, this is the same constraint as for the PIM. Thus we have shown that the constraint in the WRM is an ε-perturbation of the constraint in the PIM.

§ CONCLUSION

The transfer operator of the WRM is a perturbation of the transfer operator of the PIM. For the PIM we found that the free energy equals p^*, with p^* the unique solution of the equation

$$\lambda(p, 0) = 1.$$

Recalling that the average number of disks touching the boundary of the critical droplet in the WRM is $2\pi G_\kappa \beta^{1/3}$, we get that the surface free energy of the WRM equals

$$\Psi(\kappa) = 2\pi G_\kappa p^*.$$

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