Exercise: Lecture 2

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1 Exercise: Key formula linking metastability and potential theory

In this exercise you will prove a key formula for the potential-theoretic approach to metastability, namely, the link between the mean metastable crossover time and the capacity. This formula was first exploited in Bovier, Eckhoff, Gayrard and Klein [1].

1.1 Notation and potential theory

This section collects key tools from the potential-theoretic approach to metastability, which was introduced in the first lecture (see slides Introduction). For a more complete background, see Bovier and den Hollander [2, Chapters 7–8].

Consider a *reversible* discrete-time Markov process $\{\sigma_t\}_{t\geq 0}$ on a countable state space $\mathcal{S}$, with generator L and equilibrium measure μ. We denote by $\mathbb{P}_\sigma$ the law of the process conditioned on $\sigma_0 = \sigma$.

Definition 1.1 (Capacity)**.** *The* capacity *between two non-empty disjoint subsets $A, B \subset \mathcal{S}$ is defined as*

$$\mathrm{cap}(A,B) = \sum_{\sigma\in A} \mu(\sigma) e_{AB}(\sigma), \tag{1.1}$$

where e_{AB} is called the equilibrium measure *defined as*

$$e_{AB}(\sigma) = -(Lh_{AB})(\sigma), \qquad \forall \sigma \in A. \tag{1.2}$$

The function h_{AB} is the harmonic function *and is the solution of the so-called Dirichlet problem:*

$$\begin{aligned} (-Lh_{AB})(\sigma) &= 0, \quad \sigma \in \mathcal{S}\setminus(A\cup B),\\ h_{AB}(\sigma) &= 1, \quad \sigma \in A,\\ h_{AB}(\sigma) &= 0, \quad \sigma\in B. \end{aligned} \tag{1.3}$$

A fundamental relation that links Markov processes theory with potential theory is the following probabilistic interpretation of the harmonic function and the equilibrium measure

$$\mathbb{P}_\sigma(\tau_A < \tau_B) = \begin{cases} h_{AB}(\sigma), & \sigma \in \mathcal{S}\setminus(A\cup B),\\ e_{BA}(\sigma), & \sigma\in B, \end{cases} \tag{1.4}$$

where

$$\tau_A = \inf\{t > 0 : \sigma_t \in A\} \tag{1.5}$$

is the first hitting time of the set A. Consequently, we can rewrite the capacity defined in (1.1) as

$$\mathrm{cap}(A,B) = \sum_{\sigma\in A} \mu(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A). \tag{1.6}$$

The importance of capacities lies in the fact that they can be estimated with the help of two dual variational principles, the *Dirichlet principle* and the *Thomson principle*, and that the key formula links the capacity to the mean metastable crossover time (see slides Introduction). Before stating these, we first need to introduce the *last-exit biased distribution* on A for the transition from A to B, i.e.,

$$\nu_{A,B}(\sigma) = \frac{\mu(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A)}{\sum_{\sigma\in A}\mu(\sigma)\mathbb{P}_\sigma(\tau_B < \tau_A)}, \qquad \sigma \in A. \tag{1.7}$$

A crucial formula is the following relationship between mean hitting time and capacity,

$$\sum_{\sigma\in A}\mu(\sigma)e_{AB}(\sigma)\mathbb{E}_\sigma[\tau_B] = \sum_{\sigma'\in\mathcal{S}}\mu(\sigma')h_{AB}(\sigma'), \tag{1.8}$$

which, by (1.7), gives

$$\mathbb{E}_{\nu_{AB}}[\tau_B] = \sum_{\sigma\in A}\nu_{AB}(\sigma)\mathbb{E}_\sigma[\tau_B] = \frac{1}{\mathrm{cap}(A,B)}\sum_{\sigma'\in\mathcal{S}}\mu(\sigma')h_{AB}(\sigma'). \tag{1.9}$$

In particular, for $A = \{a\}$,

$$\boxed{\mathbb{E}_a[\tau_B] = \frac{1}{\mathrm{cap}(a,B)}\sum_{\sigma'\in\mathcal{S}\setminus B}\mu(\sigma')h_{aB}(\sigma').} \tag{1.10}$$

1.2 Exercise

(i) Prove formula (1.10).

(ii) Explain in words why in the metastable regime the numerator simplifies to the Gibbs weight of the metastable state.

1.3 Guidelines for solving the exercise

- **Step 1:** Recall that the general Dirichlet problem,

$$\begin{aligned} (-Lf)(x) &= g(x), \quad x \in \mathcal{S}\setminus B, \\ f(x) &= 0, \quad x \in B, \end{aligned} \tag{1.11}$$

  for some set B and some function f, has a unique solution that can be represented in the form

$$f(x) = \sum_{y\in\mathcal{S}\setminus B} G_B(x,y)g(y), \tag{1.12}$$

  where G_B is the Green function. We want to show that the Green function can be computed in terms of equilibrium potentials and equilibrium measures. Putting together (1.2) and (1.3), we obtain the following Dirichlet problem:

$$\begin{aligned} -(Lh_{AB})(x) &= e_{AB}(x), \quad x \in \mathcal{S}\setminus B, \\ h_{AB}(x) &= 0, \quad \sigma \in B. \end{aligned} \tag{1.13}$$

  Write its solution $h_{AB}(x)$, $x \in \mathcal{S}$, in terms of the Green function and the equilibrium measure.

- **Step 2:** Since the dynamics is assumed to be reversible, we have

$$\mu(x)G_B(x,y) = \mu(y)G_B(y,x), \qquad \forall x,y \in \mathcal{S}. \tag{1.14}$$

  Use this fact and Step 1 to rewrite $h_{AB}(x)$, $x \in \mathcal{S}$.

- **Step 3:** Compute $\sum_{x\in\mathcal{S}}\mu(x)g(x)h_{AB}(x)$ using the expression for $h_{AB}(x)$ obtained in Step 2, where g is any function.

- **Step 4:** Suppose that f is a solution of (1.11). Again use the expression (1.12) in the previous equation in order to have f appearing in one side.

- **Step 5:** Suppose that $A = \{a\}$ is a singleton. Use the fact that when $g \equiv 1$, $f(x) = \mathbb{E}_x[\tau_B]$ for $x \in \mathcal{S}\setminus B$ and $f(x) = 0$ for $x \in B$.

References

[1] A. Bovier, M. Eckhoff, V. Gayrard and M. Klein. *Metastability in stochastic dynamics of disordered mean-field models*, Probab. Theory Related Fields, Vol. 119, 99–161, 2001.

[2] A. Bovier and F. den Hollander. *Metastability: A Potential-Theoretic Approach*, Grundlehren der Mathematischen Wissenschaften, Vol. 351. Springer, 2015.