Mini Course on Asymptotics of Interacting Stochastic Processes on Sparse Graphs
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Exercise Set 2

Notation and Terminology

We will use the same notation used in the lectures. In addition, we use the following:

- Given random elements Y, Y' and Z on the same probability space, we will write $Y \perp Y'$ to mean Y is independent of Y', and we will write $Y \perp Y'|Z$ to mean Y is conditionally independent of Y', given Z.
- Also given vertices v, u in a graph H, $d_H(u,v)$ denotes the graph distance between u and v.
- If $G = (V, E)$ is a graph, and $V' subset V$, then the induced subgraph of G on V' is the graph $G = (V', E')$ with

$$E' = \{(u,v) \in E : u, v \in V'\}.$$

Definition 2.9. Given a connected graph G, $S \subset V$ is said to be a vertex cut-set of G if the removal of S and any edges incident to it disconnects the graph, i.e., the graph

$$\tilde{G} := (V \setminus S, \tilde{E}), \quad \tilde{E} := \{(u,v) \in E : u, v \in V \setminus S\},$$

is disconnected (i.e., has more than one connected component).

Definition 2.10. Given a graph G and measurable space $\mathcal{X}$, $\mu \in \mathcal{P}(\mathcal{X}^V)$ is said to be a Markov random field (MRF) if for every finite $A \subset G$, and $B \subset V \setminus [A \cap \partial A]$,

$$\mu_A \perp \mu_B | \mu_{\partial A}.$$

Problems. The first two problems relate to Lec. 2; the rest are relevant for Lec. 3.

(1) Show that given a sequence of finite graphs $|G_n| \to \infty$, if $(G_n, \mathbf{X}^{\mathbf{G_n}, \mathbf{x^n}})$ converges in probability in the local weak sense to $(G, \mathbf{X}^{\mathbf{G},\mathbf{x}})$ then μ^{G_n, x^n} converges weakly to $\mu^{G,x}$.

(2) Consider finite random graphs G_n such that G_n converges in probability in the local weak sense to a random element G of $\mathcal{G}_*$, and $|G_n| \to \infty$ in probability. Let U_1^n and U_2^n be two independent vertices that are both chosen uniformly at random from G_n, conditional on the graph structure G_n, then show that $d_{G_n}(U_1^n, U_2^n) \to \infty$ as $n \to \infty$.

(3) For $i = 1, 2$, Let $\mathcal{X}_i$, be a measurable space, and let X_i be a random element of $\mathcal{X}_i$. Then, given $X_1 \perp X_2 | X_3$, for any measurable functions $\varphi : \mathcal{X}_1 \times \mathcal{X}_3 \to \mathbb{R}$ and $\psi : \mathcal{X}_2 \times \mathcal{X}_3 \to \mathbb{R}$.

(a) Show that

$$(X_1, \varphi(X_1, X_3)) \perp (X_2, \psi(X_2, X_3)) | X_3$$

(b) Show that

$$X_1 \perp X_2 | (X_2, \varphi(X_1, X_3), \psi(X_2, X_3)).$$

(4) Suppose $X, Y, X_n, Y_n, n \in \mathbb{N}$, are random elements of a measurable space $\mathcal{X}$, say defined on a common probability space, such that (X_n, Y_n) converges weakly to (X, Y) as $n \to \infty$. Consider the following statement: for any bounded measurable functions $f, g : \mathcal{X} \to \mathbb{R}$, as $n \to \infty$,

$$\mathbb{E}[f(X_n)|g(Y_n)] \to \mathbb{E}[f(X)|g(Y)].$$

State whether this statement is true or false, by either providing a rigorous proof or a counterexample.

(5) In both parts below, suppose $G = (V, E)$ is a simple graph, $\mathcal{X}$ is a measurable space and μ lies in $\mathcal{P}(\mathcal{X}^V)$.

(a) Suppose that G is finite and for every vertex cut-set S of G and associated disjoint components A and B such that A, B, S is a partition of V,

$$\mu_A \perp \mu_B | \mu_S.$$

Then show that μ is an MRF with respect to G. Is this also true if G is infinite?

(b) Suppose that G is a locally finite, infinite graph and μ is an MRF with respect to G.[1], Then, is it true that

$$\mu_A \perp \mu_B | \mu_{\partial A}$$

for any infinite set $A \subset V$ and $B \subset V \setminus [A \cup \partial A]$? Justify your answer with a rigorous proof or a counterexample.

(6) Suppose $G = (V, E)$ is a locally finite infinite graph, $\mathcal{X}$ is a measurable space and μ is an MRF on $\mathcal{X}^V$ with respect to G. Given any subset $V' \subset V$, is $\mu|_{V'}$, the restriction of μ to V', an MRF with respect to the induced subgraph G' of G on V'. Provide a rigorous justification for your answer.

[1]For this part, for simplicity, you may assume $G = \mathbb{Z}$, the infinite 2-tree, and $\mathcal{X}$ is a finite set.