Mini Course on Asymptotics of Interacting Stochastic Processes on Sparse Graphs

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Exercise Set 1

We first introduce some common notation and definitions used in this exercise sheet.

- Given any Polish space $\mathcal{Y}$, we use $\Rightarrow$ for convergence in distribution, and $\overset{(p)}{\to}$ for convergence in probability, of $\mathcal{Y}$-valued random elements (defined on a common probability space).
- Given any Polish space $\mathcal{Y}$, let $\mathcal{C}_b(\mathcal{Y})$ be the space of bounded continuous functions on $\mathcal{Y}$.

Definition 2.8. Given any (possibly disconnected) graph $H = (V_H, E_H)$, and $v \in V_H$, define $\mathsf{C}_v(H)$ to be (the equivalence class in $\mathcal{G}^*$ of) the connected component of H that contains v, rooted at v.

Problems.

(1) Show that if $\mathcal{Y}$ is complete and separable, then so is $\mathcal{G}_*[\mathcal{Y}]$.

(2) Given a Polish space $\mathcal{Y}$, show that the root map
$$\mathcal{G}_*[\mathcal{Y}] \ni (G, \rho, y) \to y_\rho \in \mathcal{Y}$$
is continuous.

(3) Suppose $\mathcal{Y}$ is a complete separable metric space. Let (G, y), $(G_n, y^n) \in \mathcal{G}_*[\mathcal{Y}]$ and assume G_n and G are finite. Define the empirical measure
$$\mu^G := \frac{1}{|G|} \sum_{v \in G} \delta_{y_v}, \qquad \mu^{G_n} := \frac{1}{|G_n|} \sum_{v \in G_n} \delta_{y_v}.$$
If $(G_n, y^n) \to (G, y)$ in $\mathcal{G}_*[\mathcal{Y}]$, then show that $\mu^{G_n} \to \mu^G$ in $\mathcal{P}(\mathcal{Y})$.

(4) Given a sequence of (non-empty) random graphs (G_n) and a random element G of $\mathcal{G}_*$, show that
$$\frac{1}{|G_n|} \sum_{v \in G_n} f(\mathsf{C}_v(G_n)) \quad \overset{(p)}{\to} \quad \mathbb{E}[f(G)], \qquad \forall f \in \mathcal{C}_b(\mathcal{G}_*), \tag{2.11}$$
if and only if
$$\frac{1}{|G_n|} \sum_{v \in G_n} \delta_{\mathsf{C}_v(G_n)} \quad \overset{(p)}{\to} \quad \mathrm{Law}(G) \qquad \text{in } \mathcal{P}(\mathcal{G}_*), \forall f \in \mathcal{C}_b(\mathcal{G}_*),$$

(5) Find an example of a sequence of (possibly disconnected) random graphs $(G_n), G$, such that
$$\mathbb{E}\left[\frac{1}{|G_n|} \sum_{v \in G_n} \delta_{\mathsf{C}_v(G_n)}\right] \quad \to \quad \mathbb{E}\left[f(G)\right], \qquad \forall f \in \mathcal{C}_b(\mathcal{G}_*),$$
but (2.11) does not hold.

(6) Let $\mathcal{G}(n, p_n)$ denote the Erdös-Rényi graph with n vertices and edge probabilities being iid with probability p_n.

(a) Show that
$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{G}(n, p_n) \text{ is disconnected }\right) = \begin{cases} 1 & \text{if } \lim_{n\to\infty} \dfrac{p_n}{n \log n} = \infty \\ 0 & \text{if } \lim_{n\to\infty} \dfrac{p_n}{n \log n} = 0. \end{cases} \tag{2.12}$$

In other words $\ln n/n$ is a sharp threshold for the connectedness of $\mathcal{G}(n, p_n)$.