Recap of Lecture 1

$$\mathbf{X_v^{G,x}(t+1) = F_v\left(X_v^{G,x}(t), X_{N_v}^{G,x}(t), \xi_v(t+1)\right)},$$

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A1: Theorem 1: Lacker-R-Wu; '19/'20

if (G_n, x^n) converges locally to (G, x) in distribution, then $(G_n, \mathbf{X^{G_n,x^n}})$ converges locally in distribution to $(G, \mathbf{X^{G,x}})$

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Proof: Markov chains. Simple inductive argument.

Diffusions. A more involved coupling argument.

Cont. time Markov chains. More subtle. When maximal degree of G is not uniformly bounded, even well-posedness of $\mathbf{X}^{G,x}$ is not immediate (**Ganguly-R '21**).

Recap of Lecture 2

$$\mu^{G,x} := \frac{1}{|G|}\sum_{v\in G}\delta_{X_v^{G,x}} \qquad \mu^{G,x}(t) := \frac{1}{|G|}\sum_{v\in G}\delta_{X_v^{G,x}(t)}$$

Q2. Do the global empirical measures μ^{G_n,x^n} converge?
Is the limit a deterministic measure? If so, is it $\mathrm{Law}(\mathbf{X}_\rho^{\mathbf{G},\mathbf{x}})$?

Theorem 2: Lacker-R-Wu; '19/'20

Suppose (G_n, x^n) converges in probability in the local weak sense to (G, x). Then $(G_n, \mathbf{X}^{\mathbf{G_n},\mathbf{x^n}})$ converges in probability in the local weak sense to $(G, \mathbf{X}^{\mathbf{G},\mathbf{x}})$ and hence, $\mu^{G_n,x^n} \Rightarrow \mathrm{Law}(\mathbf{X}_\rho^{\mathbf{G},\mathbf{x}})$.
Example: $G_n = \mathcal{G}(n, p_n)$ with $np_n \to c$; x^n i.i.d. init. cond.

Theorem 3 Lacker-R-Wu; '19/'20

But the limit of μ^{G_n,x^n} could be stochastic when (G_n, x^n) converges only in law in the local weak sense to (G, x).

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Theorem 3 Lacker-R-Wu; '19/'20

But the limit of μ^{G_n,x^n} could be stochastic when (G_n, x^n) converges only in law in the local weak sense to (G, x).

However, in many other cases, the limit could be deterministic but not coincide with $\mathrm{Law}(\mathbf{X}_\rho^{\mathbf{G},\mathbf{x}})$!!

References related to Lectures 1 and 2

Local Weak Convergence of Stochastic Processes on Sparse Graphs

• Oliveira, Reis, Stolerman, "Interacting diffusions on sparse graphs: hydrodynamics from local weak limits," *EJP* **25** (2020).

Local Weak Convergence & Convergence of Empirical Measures of Stochastic Processes on Sparse Graphs

• Lacker, R., Wu, "Large sparse networks of interacting diffusions," Arxiv Preprint (2019)

• Lacker, R., Wu, "Local weak convergence for sparse networks of interacting processes," Arxiv Preprint (2020)

• Ganguly and R., "Limits of empirical measures of interacting particle systems on large sparse graphs," *near completion*, (2021)

Related literature for static models

- Aldous and Steele, Probabilistic Combinatorial Optimization and Local Weak Convergence, Probability on discrete structures, 1-72 **382**, 2004.
- A. Dembo and A. Montanari, Gibbs Measures and Phase Transitions on Sparse Random Graphs, *Braz. J. Probab. Stat.* **24** (2):137-211 (2010).
- Numerous other papers by C. Bordenave, M. Lelarge, N. Litvak, M. Olvera-Craviato, J. Salez, R. van der Hofstad, ...

Lecture 3

The Main Question: Characterizing Marginal Dynamics

In Lectures 1 and 2 we have shown:

if $(G_n, x^n) \to (G, x)$ in probability in the local weak sense, then

$$\mathbf{X}_{\rho_\mathbf{n}}^{\mathbf{G_n},\mathbf{x^n}} \Rightarrow \mathbf{X}_{\rho}^{\mathbf{G},\mathbf{x}} \qquad \text{and} \qquad \mu_{\rho_n}^{G_n,x^n} \Rightarrow \mathrm{Law}(\mathbf{X}_{\rho}^{\mathbf{G},\mathbf{x}})$$

... which motivates us to ask:

Q3. can one autonomously characterize the marginal dynamics of a fixed or "typical particle" $\mathbf{X}_{\rho}^{\mathbf{G},\mathbf{x}}(\mathbf{t})$, $t \in [0, T]$?

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Outline of Lecture 3

Conditional Independence Properties of the Infinite System

Derivation of Marginal Dynamics

Implications of the Results

Conditional Independence Properties of the Infinite System

A Static Analog: Markov Random Fields

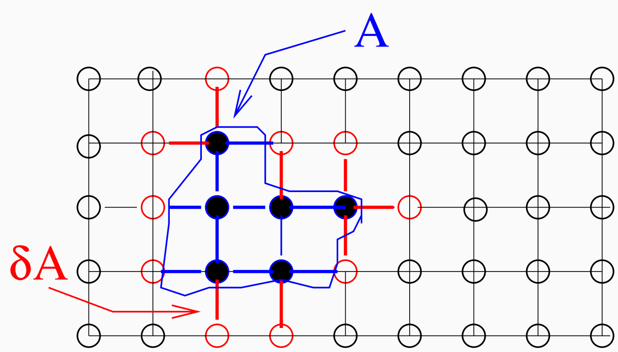
State space $\mathbb{S}$; $\{Y_v, v \in V\}$ canonical variables acting on $\mathbb{S}^V$

Defn. A probability measure π on $\mathbb{S}^V$ is said to be a Markov Random Field (MRF) wrt $G = (V, E)$ if for π a.e. η_A,

$$\pi\left(Y_A = \eta_A | Y_{V\setminus A} = \eta_{V\setminus A}\right) = \pi\left(Y_A = \eta^A | Y_{\partial A} = \eta_{\partial A}\right)$$

where ∂A is the boundary of A:

$$\partial A = \{u \in V\setminus A : \exists u \in A \text{ s.t. } u \sim v\},$$



Examples: product meas, Ising model, Potts model, hard core model, Gibbs measures, ...

Markov Random Fields

An Equivalent Formulation: $(Y_v)_{v \in V}$ is a MRF on S^V wrt $G = (V, E)$ if for finite $A \subset V$, $B \subset V \setminus [A \cup \partial A]$,

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \ \mid \ (Y_v)_{v \in \partial A},$$

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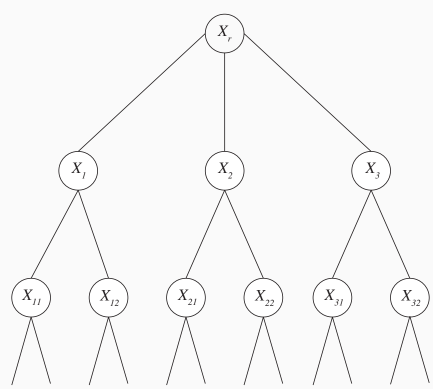
When G is a tree



Fact: Tree structure allows one to more easily analyze the marginal distribution at a node of a MRF

In Search of a Conditional Independence Property

Fix (G, x) infinite. Denote $\mathbf{X} = \mathbf{X}^{\mathbf{G},\mathbf{x}}$. Set $\sigma = I$, $(\mathbf{X_v}(\mathbf{0}))_{v \in V}$ iid.

$$\mathbf{X_v}(\mathbf{t}+\mathbf{1}) = \mathbf{F}\left(\mathbf{X_v}(\mathbf{t}), \mathbf{X_{N_v}}(\mathbf{t}), \xi_\mathbf{v}(\mathbf{t}+\mathbf{1})\right),$$

$$\mathbf{dX_v}(\mathbf{t}) = \mathbf{b}(\mathbf{X_v}(\mathbf{t}), \mathbf{X_{N_v}}(\mathbf{t}))\mathbf{dt} + \mathbf{dW_v}(\mathbf{t}).$$

Question A:

For $t > 0$, will $(\mathbf{X_v}(\mathbf{t}))_{v \in V}$ form a MRF wrt G?

In other words, for finite $A \subset V$ and $B \subset V \setminus [A \cup \partial A]$,

Is $\mathbf{X_A}(\mathbf{t}) \perp \mathbf{X_B}(\mathbf{t}) | \mathbf{X_{\partial A}}(\mathbf{t})$?

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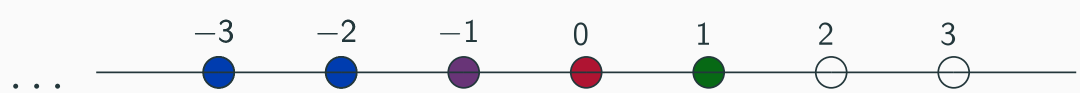
$G = \mathbb{Z}, A = \{-1, -2, \ldots, -10\}, A' = \{-1\} \subset A, \partial A = \{0, -11\}, B = \{1$



$\ldots t = 4$

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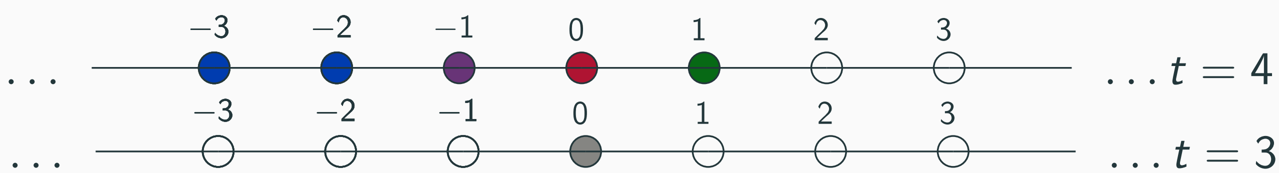

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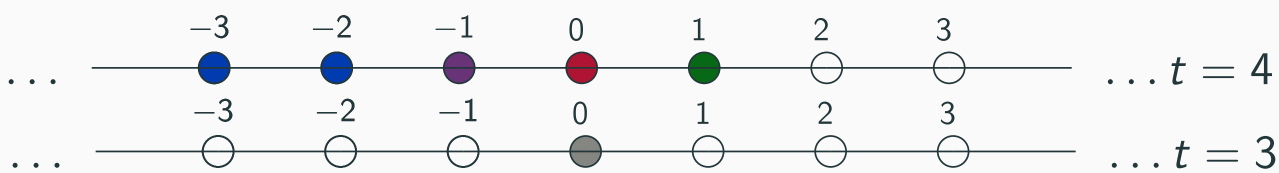

Answer A:

No!

In Search of a Conditional Independence Property

$$\mathbf{X_v(t+1) = F\,(X_v(t), X_{N_v}(t), \xi_v(t+1))},$$

$$\mathbf{dX_v(t) = b(X_v(t), X_{N_v}(t))dt + dW_v(t)}.$$

Question B:

For $t > 0$, do the particle histories $(X^v[t])_{v \in V}$ form a MRF wrt G?

Henceforth, $x[t] := (x(s), s \in [0, t])$.

In Search of a Conditional Independence Property

$$\mathbf{X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right)},$$

Reformulation of Question B:

Given $t > 0$, for any finite $A \subset V$ and $B \subset V \setminus [A \cup \partial A]$,

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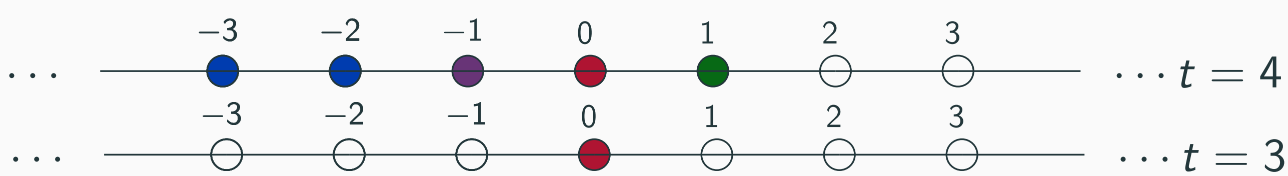

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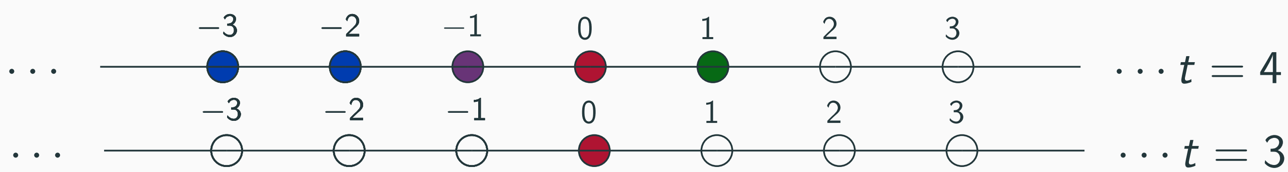


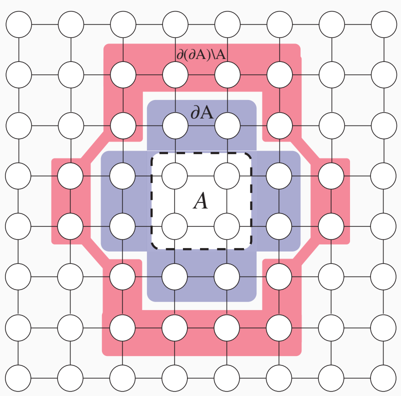
Answer B:

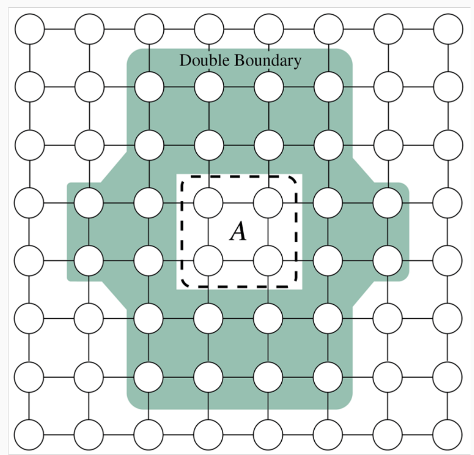
No!

Second-order Markov Random Fields

Double Boundary

$$\partial^2 A = \partial A \cup [\partial(\partial A) \setminus A]$$





Definition: A family of random variables $(Y^v)_{v \in V}$ is a 2nd-order Markov random field if

$$Y_A \perp Y_B \ \mid \ Y_{\partial^2 A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Trying again ...

$$\mathbf{X_v(t+1)} = \mathbf{F}\left(\mathbf{X_v(t)}, \mathbf{X_{N_v}(t)}, \xi_\mathbf{v}(\mathbf{t+1})\right),$$

Question C:

Given $t > 0$, for any finite $A \subset V$ and $B \subset V \setminus [A \cup \partial^2 A]$, is

$$\mathbf{X_A[t]} \perp \mathbf{X_B[t]} | \mathbf{X_{\partial^2 A}[t]}?$$

Trying again ...

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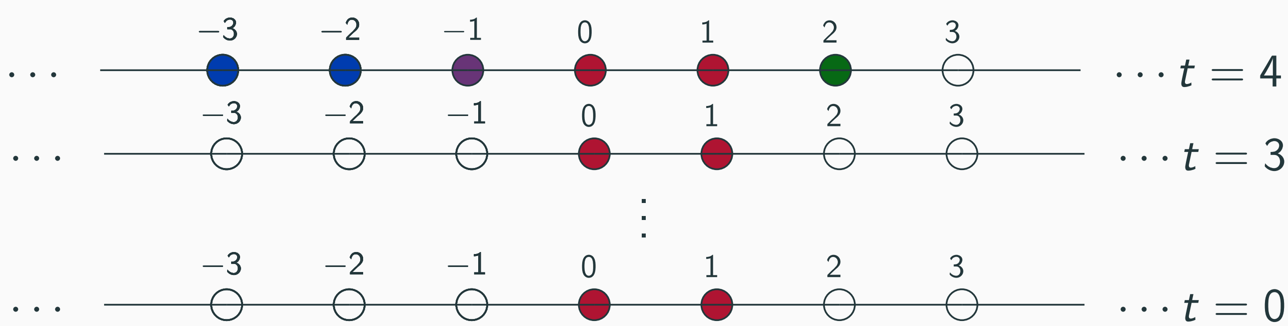

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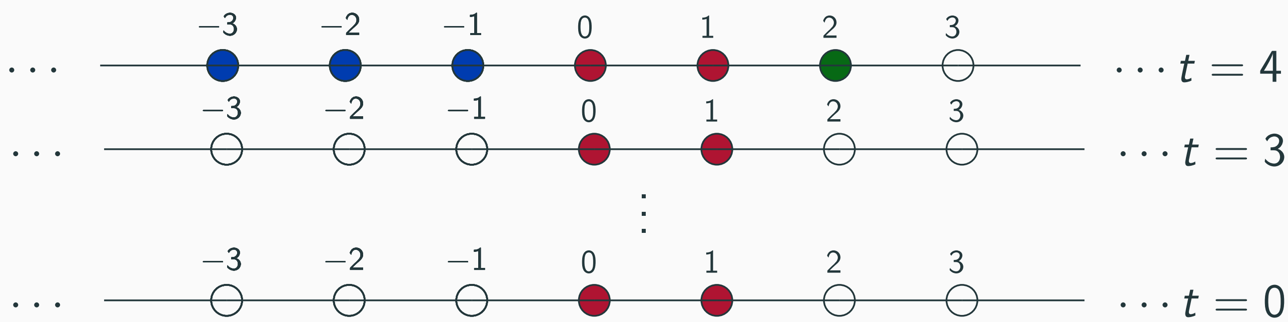

Theorem 4: (Lacker, R, Wu '18, Ganguly-R '21) YES!

$$\mathbf{X_A[t]} \perp \mathbf{X_B[t]} | \mathbf{X_{\partial^2 A}[t]}$$

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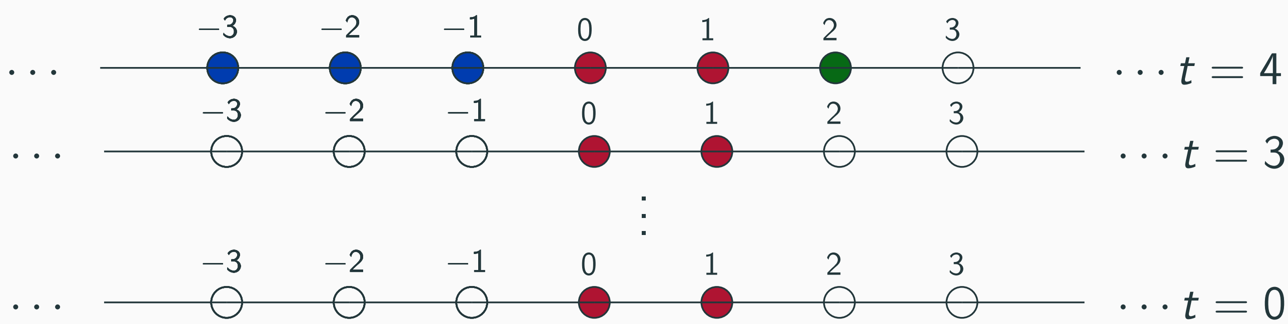

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Generalizations: In fact,

- this result holds even when $(\mathbf{X_v(0)})_{\mathbf{v \in V}}$ is just **a second-order MRF** – do not require $(X_v(0))_{v \in V}$ i.i.d.

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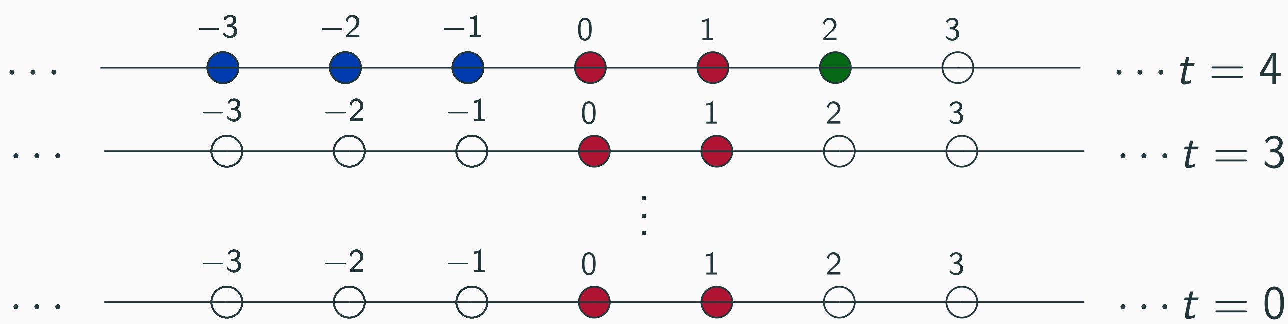

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- this result holds even when $(\mathbf{X_v(0)})_{\mathbf{v \in V}}$ is just **a second-order MRF** – do not require $(X_v(0))_{v \in V}$ i.i.d.
- Further, can allow A to be **infinite** (non-trivial for diffusions)

Comments on the Conditional Independence Property

Some related Work: mostly for $G = \mathbb{Z}^d$

- For **gradient** diffusions on $\mathbb{Z}^d$: **Deuschel ('87)** and **Cattiaux, Roelly, Zessin ('96)**
- For non-gradient processes on $\mathbb{Z}^d$ with shift-invariant initial conditions: **Dereudre and Roelly (2017)**

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Of relevance to the study of Gibbs-non-Gibbs transitions

- **(den Hollander, Külske, Opoku, Redig, Roelly, Ruszel, van Enter, ...)**

Ideas behind the Proof

Markov Chain Setting

$$\mathbf{X_v}(\mathbf{t}+\mathbf{1}) = \mathbf{F}\left(\mathbf{X_v}(\mathbf{t}), \mathbf{X_{N_v}}(\mathbf{t}), \xi_\mathbf{v}(\mathbf{t}+\mathbf{1})\right),$$

$$\mathbf{X_A}[\mathbf{t}] \perp \mathbf{X_B}[\mathbf{t}] | \mathbf{X}_{\partial^2 \mathbf{A}}[\mathbf{t}]?$$

Proof "by hand"

1. Establish some general conditional independence relations (see Problem Set 2)
2. Use the dynamics to extract appropriate functional relations
3. Combine to get the proof

Ideas behind the Proof

Diffusion Setting

$$\mathbf{dX_v(t) = b(X_v(t), X_{N_v(G)}(t))dt + dW_v(t)}$$

Invokes the Gibbs-Markov/Hammersley-Clifford Theorem

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Definition

$\mathbb{S}$ measurable space. A nonnegative function $f : \mathbb{S}^V \mapsto \mathbb{R}_+$ is said to factor on a finite graph G if there exist functions $f_K : \mathbb{S}^K \to \mathbb{R}_+$, $K \in \mathrm{cl}(G)$, such that

$$f(x) = \prod_{K \in \mathrm{cl}(G)} f_K(x^K), \quad x \in \mathbb{S}^V. \tag{1}$$

Gibbs-Markov/Hammersley-Clifford Theorem

Version when $\mathbb{S}$ is a Discrete State Space

Gibbs-Markov/Hammersley-Clifford theorem ('70's)

Given a finite graph $G = (V, E)$, if a probability mass function f on the discrete set $\mathbb{S}^V$ factors on G, or equivalently, admits the representation

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with Z the normalization constant:

$$Z = \sum_{x \in \mathbb{S}^V} \prod_{K \in \mathrm{cl}(G)} f_K(x^K),$$

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An Immediate Extension

- Let $\mathrm{cl}_2(G)$ denote the 2-cliques of G, which are subsets of V for which the induced subgraph has diameter less than or equal to 2

Gibbs-Markov/Hammersley-Clifford theorem for 2nd order

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Proof of the Conditional Independence Property

$$\mathbf{dX_v(t) = b(X_v(t), X_{N_v(G)}(t))dt + dW_v(t)}$$

$$\mathbf{X_A[t] \perp X_B[t] | X_{\partial^2 A}[t]}$$

Main Steps of the Proof in the Diffusion Case

- On any finite graph G, use Girsanov's theorem to identify the density of the law of the SDE with respect to a certain product measure (product Wiener measure when $\sigma = I$)
- On any finite graph G, show that the density has a certain clique representation and use the 2nd-order Gibbs-Markov (Hammersley-Clifford) theorem to conclude the 2nd-order MRF property.
- Use a **subtle approximation argument** for infinite graphs G (for an idea of some subtleties, see exercise in Prob. Set 2)

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Analogous result more complicated for jump processes (Ganguly-R

Derivation of Marginal Dynamics

Marginal Dynamics

$$\mathbf{X_v}(\mathbf{t}+\mathbf{1}) = \mathbf{F}\left(\mathbf{X_v}(\mathbf{t}), \mathbf{X_{N_v}}(\mathbf{t}), \xi_{\mathbf{v}}(\mathbf{t}+\mathbf{1})\right),$$

Recall the conditional independence property

For any (not necessarrily finite) $A \subset V$, $B \subset V \setminus A \cup \partial^2 A$,

$$\mathbf{X_A}[\mathbf{t}] \perp \mathbf{X_B}[\mathbf{t}] | \mathbf{X_{\partial^2 A}}[\mathbf{t}]$$

Marginal Dynamics

$$\mathbf{X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right)},$$

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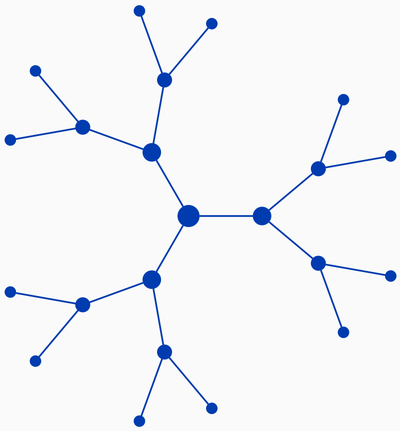
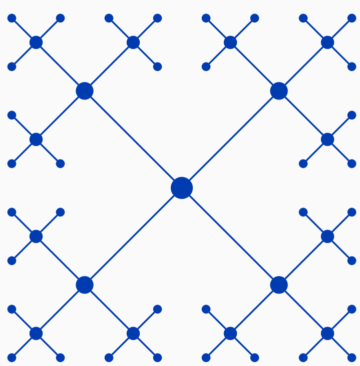
Is this conditional independence property of any use?

Marginal Dynamics on Trees

Suppose the limiting graph G is an infinite d-regular tree.



$d = 3$



$d = 4$

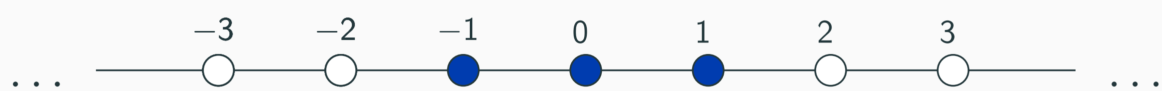
Marginal Dynamics on Trees

- Let $\mathcal{T}_\kappa$ denote the infinite κ-regular tree.
- For simplicity consider the case $\kappa = \mathbf{2}$.
- Note that **Theorem 1** implies that for a typical vertex ρ, $\{X_\rho, (X_v)_{v\sim\rho}\}$ can be obtained as the marginal of the infinite coupled system of Markov chains:

$$\mathbf{X_v(t+1) = F\left(X_v(t), X_{N_v}(t), \xi_v(t+1)\right)}, v \in \mathcal{T}_2.$$

- Identify $\mathcal{T}_2$ with $\mathbb{Z}$, set $\rho = 0$.
- Then we are interested in an autonomous characterization of the marginal law of

$$X_{-1,0,1} = (X_{-1}, X_0, X_1).$$



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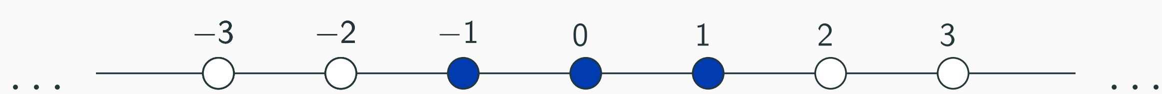
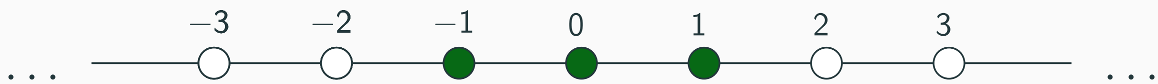


How can we exploit the conditional independence structure?

Marginal dynamics for the infinite 2-regular tree

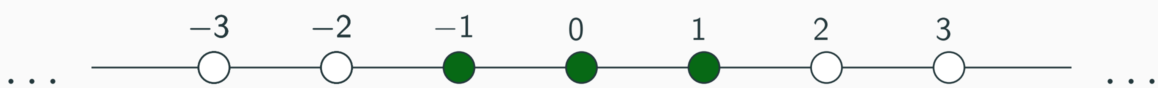


$$X_i(t+1) = F\left(X_i(t), (X_{i-1}(t), X_{i+1}(t)), \xi_i(t+1)\right), \quad i \in \mathbb{Z},$$

Goal: **Autonomous characterization of the law of $X_{-1,0,1}$**

- We will describe how to generate another stochastic process $Y = Y_{-1,0,1}$ that has the same law as the marginal process $X_{-1,0,1}$.
- but whose evolution only depends on the history of its state, and the law of the history of the state, or equivalently, the law of the marginal process $X_{-1,0,1}$, equivalently the law of $Y_{-1,0,1}$
- In other words, the dynamics of $Y_{-1,0,1}$ (is not defined for and) should make no reference to particles outside $\{-1, 0, 1\}$

Marginal dynamics for the infinite 2-regular tree

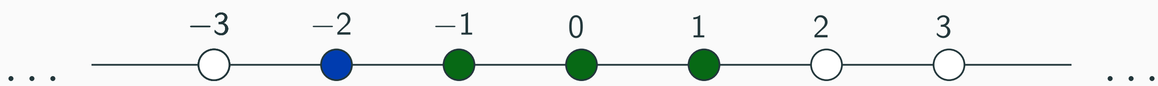


$$X_0(t+1) = F\left(X_0(t), (X_{-1}(t), X_1(t)), \xi_0(t+1)\right), \quad i \in \mathbb{Z},$$

- First, note that the evolution of the (law of the) middle particle 0 only depends on the (law of) states of the neighboring particles −1 and 1, so its evolution should exactly mimic that of X:

$$Y_0(t+1) = F\left(Y_0(t), (Y_{-1}(t), Y_1(t)), \xi_0(t+1)\right).$$

Evolution of neighboring particles



- We saw that the 0 particle evolution is simple:

$$Y_0(t+1) \quad = \quad F\left(Y_0(t), (Y_{-1}(t), Y_1(t)), \xi_0(t+1)\right).$$

- The evolution of the states of neighboring particles -1 and 1 should satisfy $(Y_{-1}, Y_1)(t+1) \stackrel{(d)}{=} (X_{-1}, X_1)(t+1)$. Recall

$$X_{-1}(t+1) = F\left(X_{-1}(t), (X_{-2}(t), X_0(t)), \xi_{-1}(t+1)\right),$$

$$X_1(t+1) = F\left(X_1(t), (X_0(t), X_2(t), \xi_1(t+1)\right)$$

- However, the law of $(X_{-1}, X_1)(t+1)$ depends on

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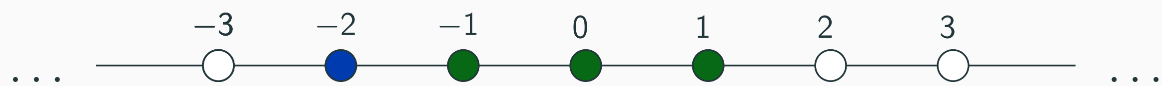

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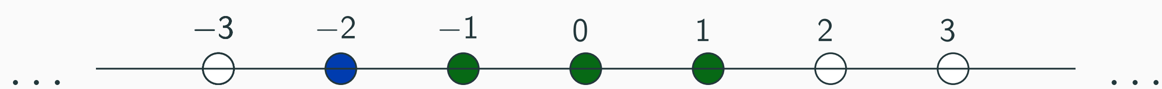
- ... which seems not obtainable from $Y_{-1,0,1}(t) \stackrel{(d)}{=} X_{-1,0,1}(t)$
... as it involves extraneous particles (X_{-2}, X_2)

Evolution of neighboring particles



$$X_{-1}(t+1) = F\left(X_{-1}(t), (X_{-2}(t), X_0(t)), \xi_{-1}(t+1)\right),$$

$$X_1(t+1) = F\left(X_1(t), (X_0(t), X_2(t), \xi_1(t+1)\right)$$

- **Key observation 1:** We have access to the law (and values) of $Y_{-1,0,1}[t]$ (or $X_{-1,0,1}[t]$), and so it suffices to know the conditional law of $(X_{-2}(t), X_2(t))$, given the past $X_{-1,0,1}[t]$.
- But can we relate this to $\mathrm{Law}(Y_{-1,0,1}[t]) = \mathrm{Law}(X_{-1,0,1}[t])$?

$$X_{-1}(t+1) = F\left(X_{-1}(t), (X_{-2}(t), X_0(t)), \xi_{-1}(t+1)\right),$$
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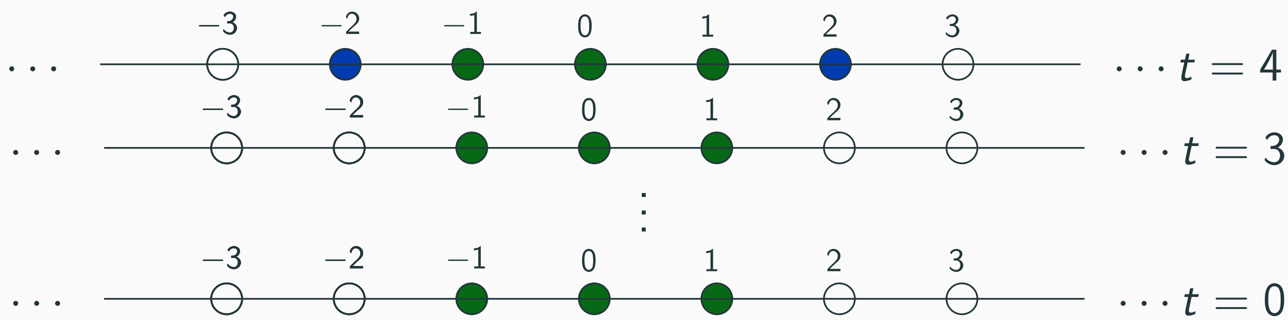
- **Recall observation 1:** It suffices to know the conditional law of $(X_{-2}(t), X_2(t))$, given the past $X_{-1,0,1}[t]$.



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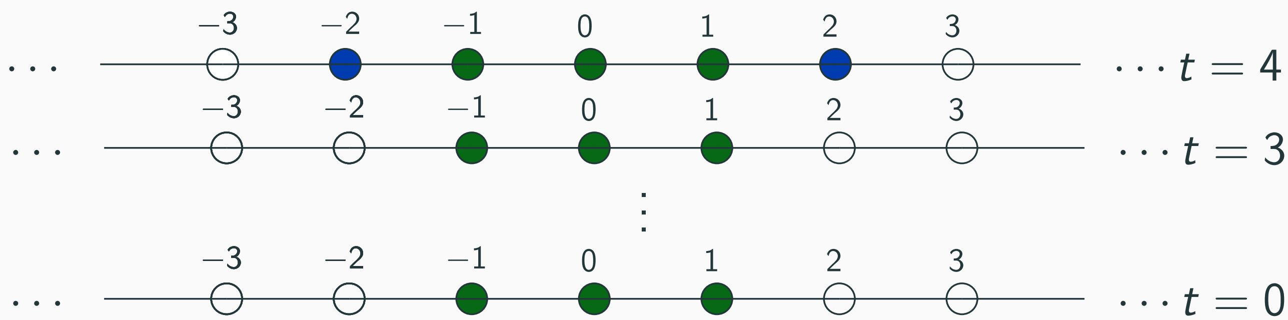


- **Key observation 2:** By the 2-MRF property of Thm 4,

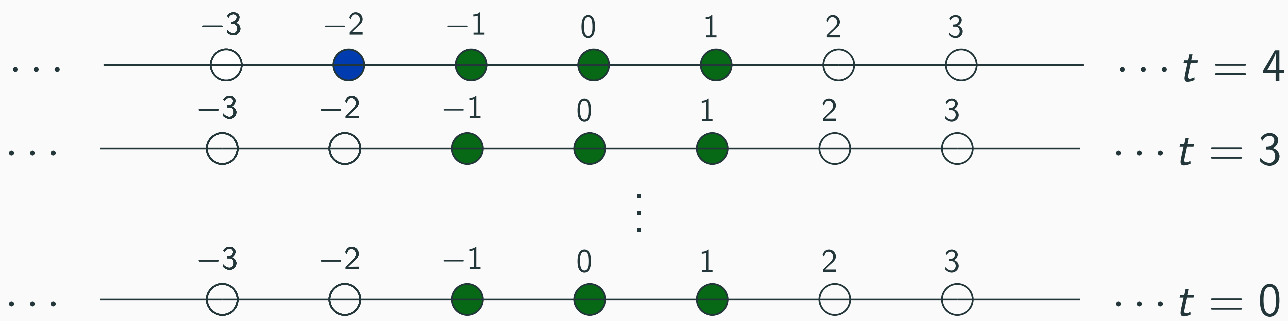
$$X_{-2}(t) \perp X_2(t) | X_{-1,0,1}[t]$$

– So it suffices to know the conditional law of $X_{-2}(t)$, given $X_{-1,0,1}[t]$

– the conditional law of $X_2(t)$, given $X_{-1,0,1}[t]$ can then be recovered by symmetry

$$X_{-1}(t+1) = F\left(X_{-1}(t), (X_{-2}(t), X_0(t)), \xi_{-1}(t+1)\right),$$

- **By Observations 1 & 2** it suffices to know the conditional law of $X_{-2}(t)$, given $X_{-1,0,1}[t]$



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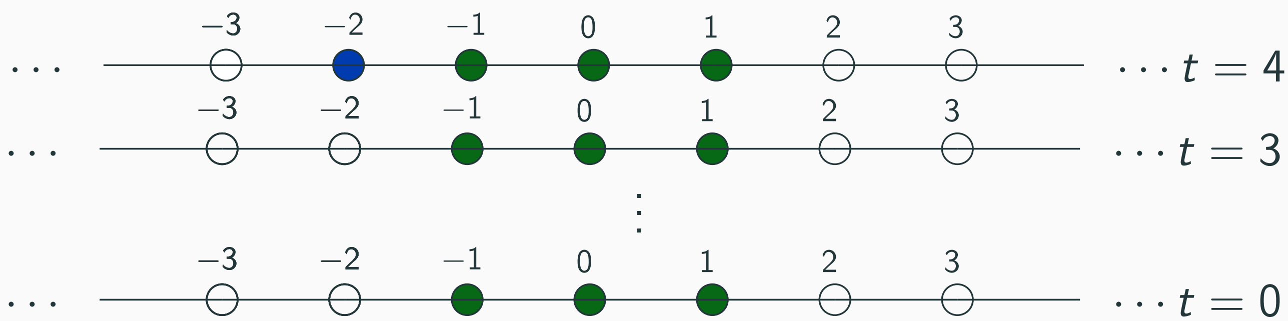
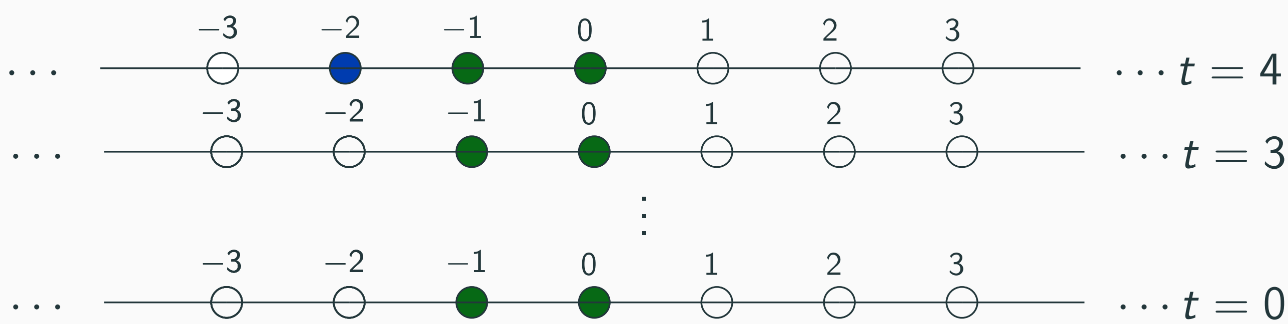
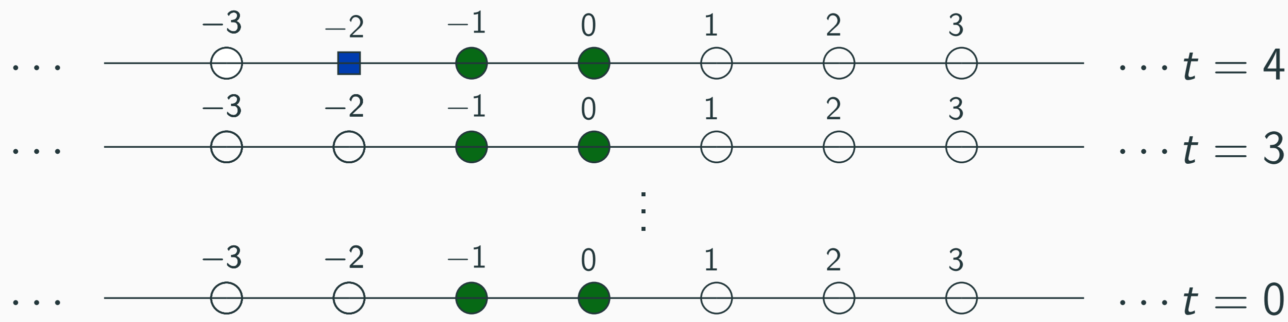


- **Key Observation 3:** By the 2-MRF property of Thm 4, this coincides with the conditional law of $X_{-2}(t)$, given $X_{-1,0}[t]$



- **Recall observations 1-3 imply:** It suffices to know the conditional law of $X_{-2}(t)$ given $X_{-1,0}[t]$.



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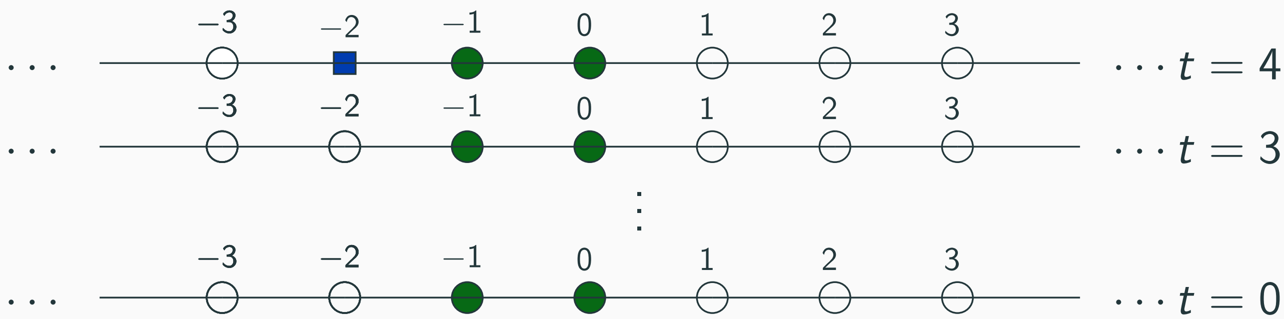

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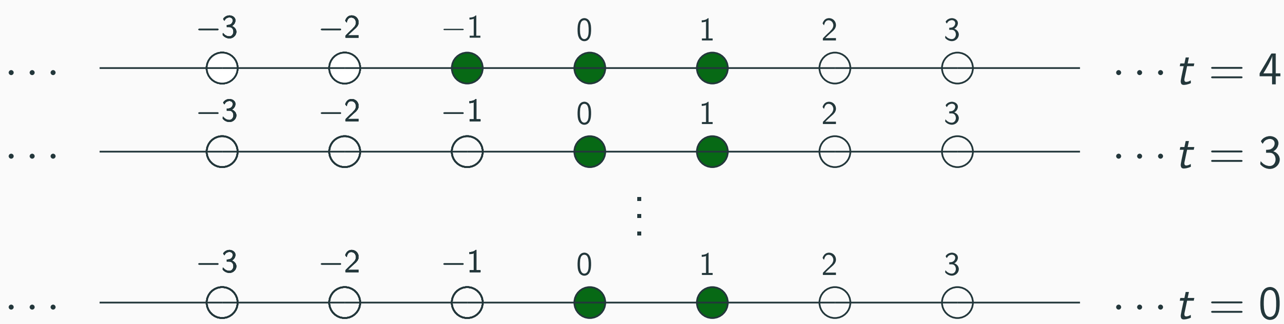
- **Key observation 4:** By translation symmetry, this is the same as the conditional law of $X_{-1}(t)$ given $X_{0,1}[t]$



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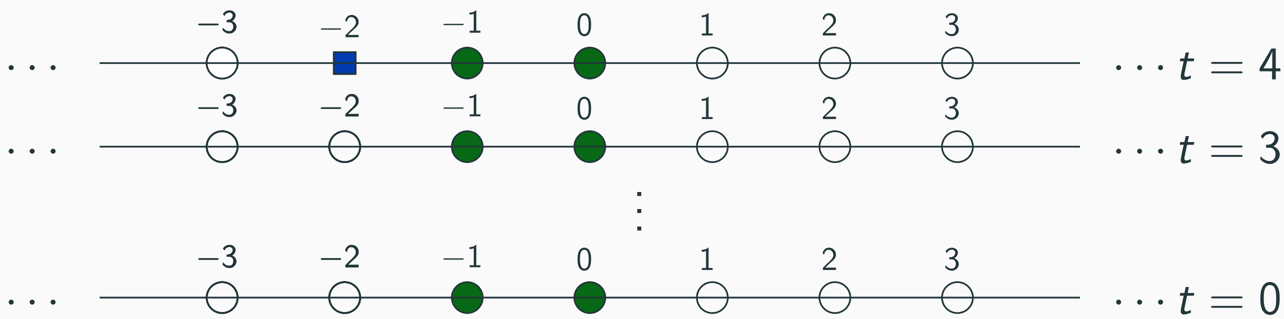

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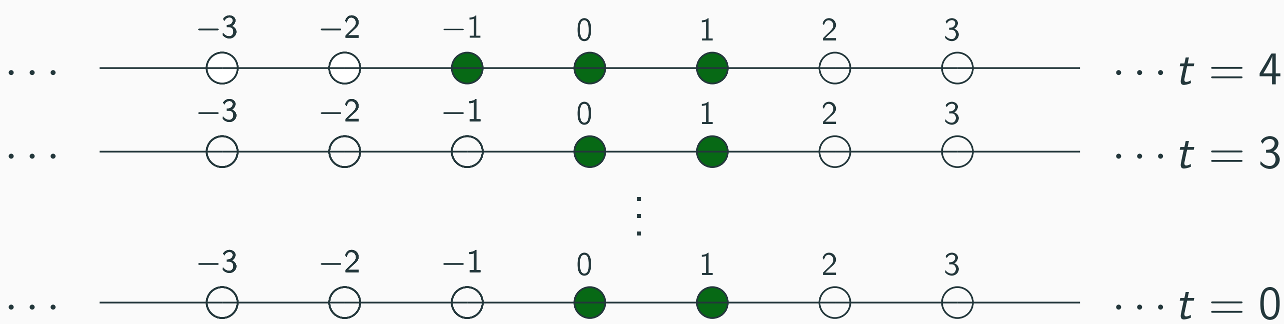

- But this only requires the knowledge of the law of $X_{-1,0,1}[t]$, (hence, of $Y_{-1,0,1}[t]$), so the **evolution is autonomous**!

Autonomous evolution of the root neighborhood

- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$

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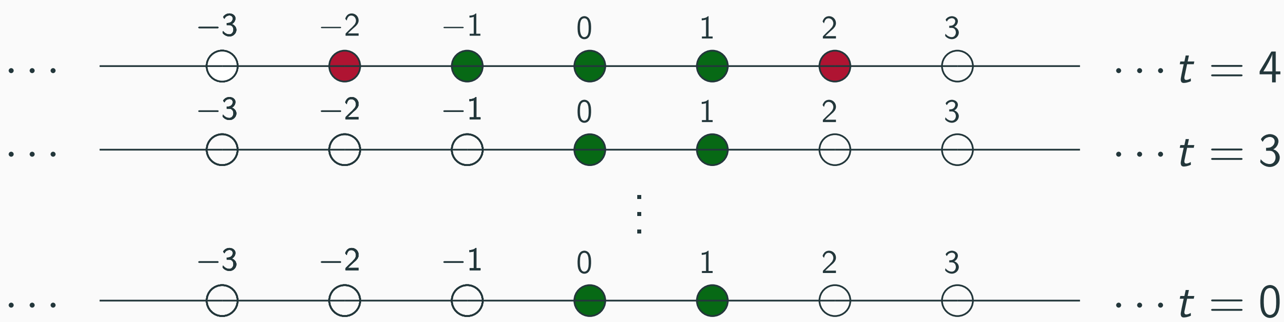
- Start with $Y_{-1,0,1}(0) = X_{-1,0,1}(0)$
- At each time $t \in \mathbb{N}_0$, define for $y_0, y_1 \in \mathcal{X}^\infty$,

$$\gamma_t(\cdot \mid y_0, y_1) = \mathrm{Law}\Big(Y_{-1}(t) \mid Y_0[t] = y_0[t], Y_1[t] = y_1[t] \Big).$$



- Sample ghost particles $Y_{-2}(t)$ andd $Y_2(t)$) so that

$$\mathbb{P}\Big(Y_{-2,2}(t) = y_{-2,2} \mid Y_{-1,0,1}[t] \Big) = \gamma_t(y_{-2} | Y_{-1,0}[t]) \gamma_t(y_2 | Y_{1,0}[t])$$

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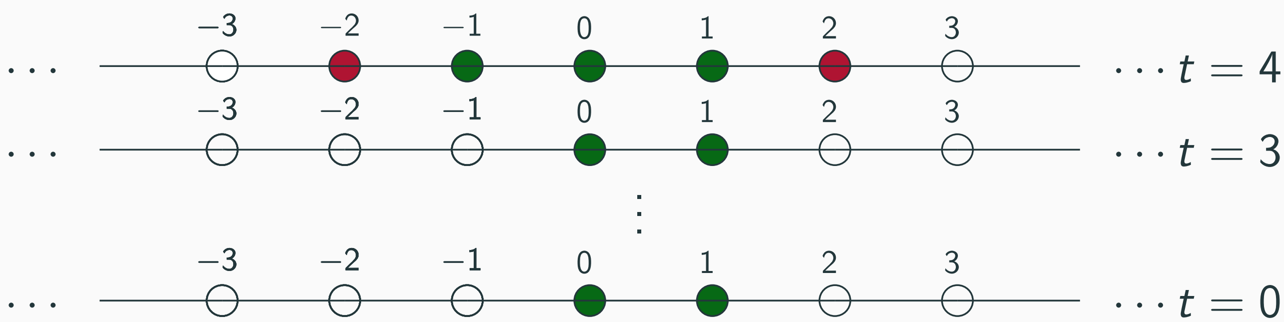

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- Sample iid noises $\xi_{-1,0,1}(t+1)$, and update:

$$Y_i(t+1) = F\Big(Y_i(t), Y_{i-1,i+1}(t), \xi_i(t+1) \Big), \quad i = -1, 0, 1$$

Structure of evolution of the root neighborhood

$$Y_{-1}(t+1) = F\Big(Y_{-1}(t), (Y_{-2}(t), Y_0(t)), \xi_{-1}(t+1)\Big),$$
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Rephrasing, without reference to "ghost particles", the evolution of the law of $Y_{-1,0,1}$ is **autonomous**, **non-Markov** and **nonlinear**:

$$Y_{-1,0,1}(t+1) = H\Big(t, Y_{-1,0,1}[t], \mathrm{Law}(\mathbf{Y_{-1,0,1}[t]}), \xi_{-1,0,1}(t+1)\Big).$$

for some measurable mapping

$$H : \mathbb{N} \times \mathcal{X}^{\infty} \times \mathcal{P}(\mathcal{X}^{\infty}) \times U \mapsto \mathcal{X}^3$$

Marginal Dynamics on the 2-regular tree: Diffusion

- As before, identify $\mathcal{T}_2 = \mathbb{Z}$, $\rho = 0$.
- Once again interested in an autonomous characterization of the marginal $X_{-1,0,1}$ of the infinite system of SDEs:

$$dX_i(t) = \frac{1}{2} \sum_{j=i+1,i-1} \bar{b}(X_i(t), X_j(t))dt + dW_i(t), \quad i \in \mathbb{Z},$$

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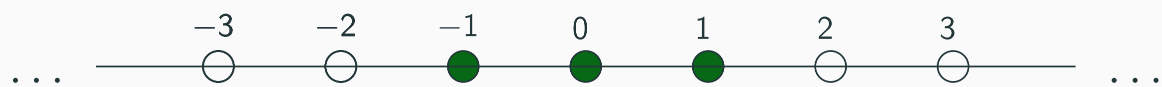
  where (for simplicity) we choose the special "linear" form of the drift
- A similar result as in the M. chain case holds, except that the derivation is much more complicated.

From Conditional Independence to Local Equations



Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$dX_i(t) = \frac{1}{2}\left(\overline{b}(X_i(t), X_{i-1}(t)) + \overline{b}(X_i(t), X_{i+1}(t))\right) dt + dW_i(t)$$

For $x_1, x_0 \in \mathcal{C}$, and $t > 0$,

$$\gamma_t(x_1, x_0) := \mathrm{Law}\big(X_{-1}(t) \,|\, X_0[t] = x_0[t],\ X_1[t] = x_1[t]\big).$$

Theorem 5 (Lacker-R-W '19): $X_{-1,0,1} \overset{d}{=} Y = (Y_{-1,0,1}$, where Y is the **unique** weak solution to

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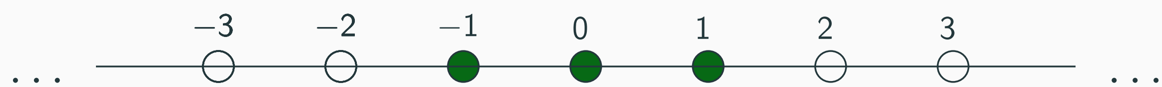

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Again, autonomous description as a **nonlinear, non-Markov** proc.

Summary: Beyond Mean-Field Limits

Mean-Field Dynamics (Dense Sequences $G_n = K_n$)

$$dX(t) = B(X(t), \mu(t))dt + dW(t), \qquad \mu(t) = \mathrm{Law}(X(t)).$$

where $B(x, m) = \int_{\mathbb{R}^d} b(x, y)\, m(dy)$.

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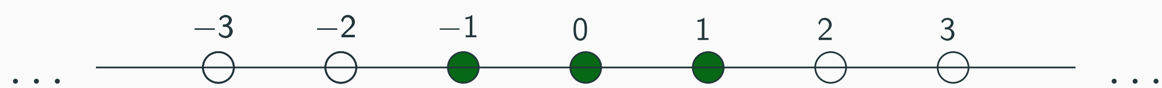

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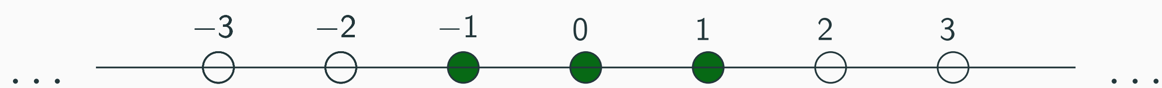

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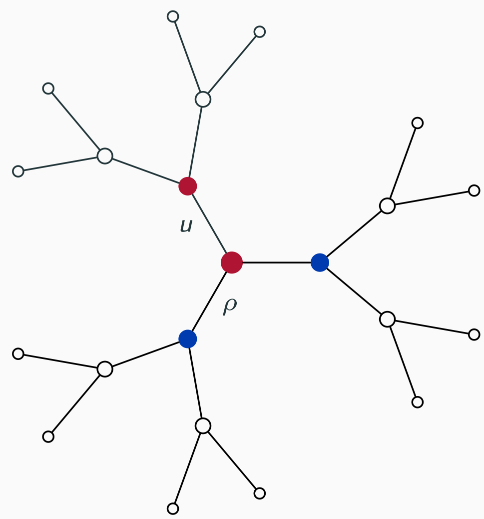
Generalizations: Infinite κ-regular trees $\mathbb{T}_\kappa$

Can derive an autonomous SDE system for root particle and its neighbors,

$$X_\rho(t), \ (X_\rho(t))_{v\sim\rho},$$

involving the conditional law of $\kappa - 1$ children given root and one other child u:

$$\mathrm{Law}((X_v)_{v\sim\rho,\, v\neq u} \,|\, X_\rho,\, X_u)$$



$\kappa = 3$

Infinite κ-regular trees

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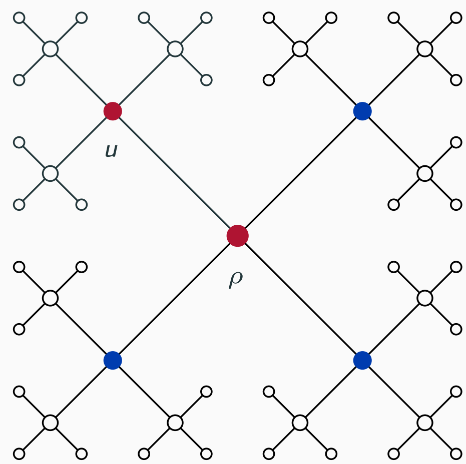

$\kappa = 4$

Infinite d-regular trees

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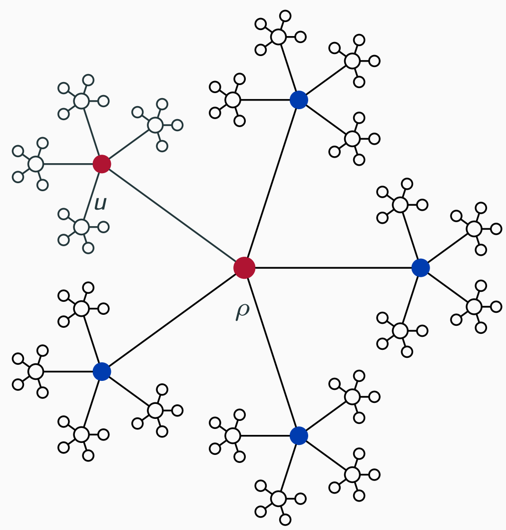

$\kappa = 5$

But what about random graph limits?

For example, can we find the marginal dynamics on a unimodular Galton-Watson tree?

Yes ... although the derivation is more complicated and now involves also averaging over the random structure of the tree s

Implications of the Results

Summary of Results

Recall that evolution of the law of $Y_{-1,0,1}$ is **autonomous**, **non-Markov** and **nonlinear**

Markov chain:

$$Y_{-1,0,1}(t+1) = H\Big(t, Y_{-1,0,1}[t], \mathrm{Law}(\mathbf{Y_{-1,0,1}[t]}), \xi_{-1,0,1}(t+1)\Big).$$

for some measurable mapping

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Diffusion:

$$dY_{-1}(t) = \tfrac{1}{2}\left(\overline{b}(Y_{-1,0}(t)) + \langle \gamma_t(Y_{-1}, Y_0), \overline{b}(Y_{-1}(t), \cdot)\rangle\right) dt + d\tilde{W}_1(t)$$

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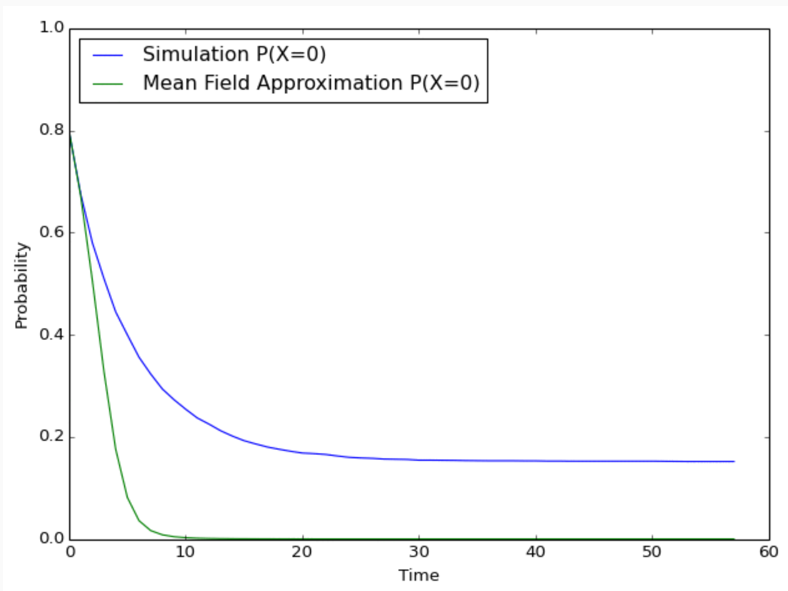
How well do these approximations perform?

Discrete-time SIR process on the cycle graph

Comparing mean-field and full simulation



Plot of probability of being healthy vs. time

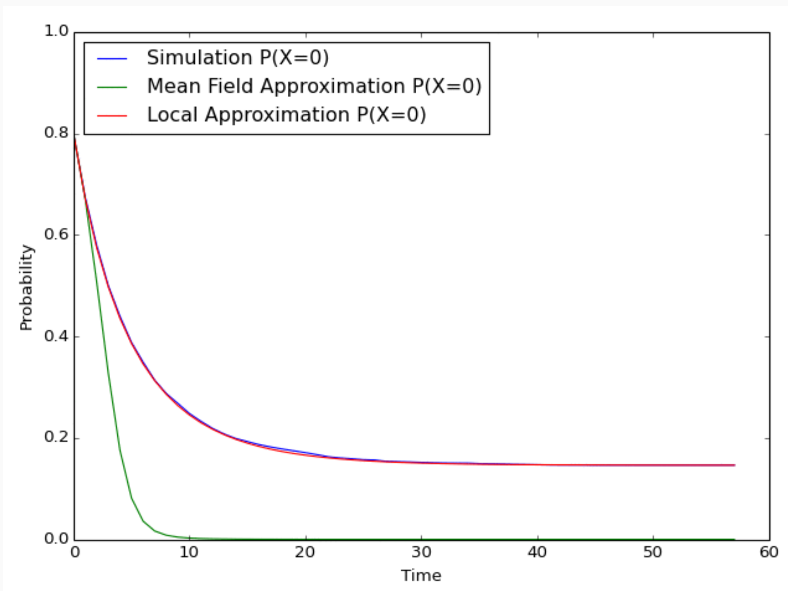
simulations due to Mitchell Wortsman

Mean-field approximation fails!

How well does the local eqn. approximation perform?

Discrete-time SIR process on the cycle graph

Comparing mean-field, full simulation and local equation



Plot of probability of being healthy vs. time

simulations due to Mitchell Wortsman

Local equation approximation works well!!

Similar observation for other processes

The Discrete-Time Contact Process

$$X_v(t+1) = F\left(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)\right),$$

State space $S = \{0,1\} = \{\text{healthy}, \text{infected}\}$. Parameters $p, q \in [0,1]$.

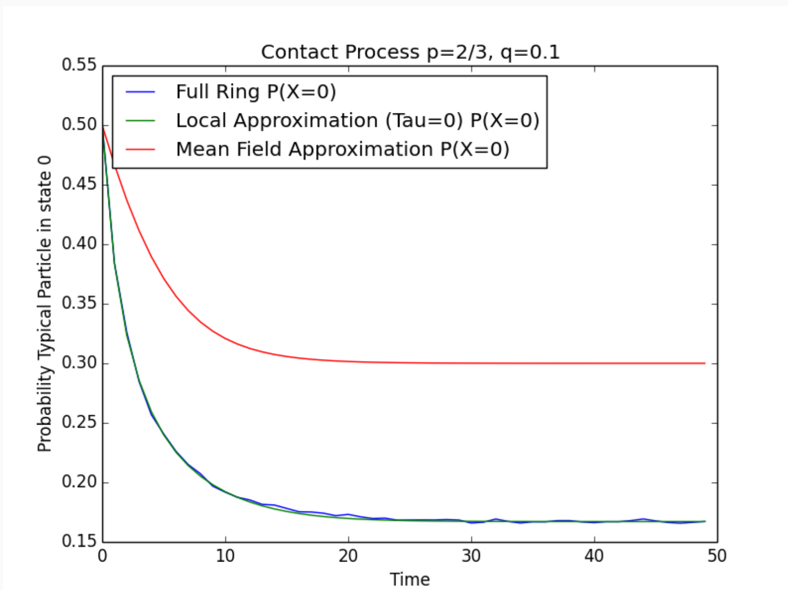
Transition rule F: At time t, if particle v is at...

- state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p. q,
- state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{p}{d_v} \sum_{u\sim v} X_u(t),$$

where $d_v =$ degree of vertex v.

Numerical Results for the Discrete-time Contact Process



Summary of the Course

- Novel characterization of asymptotic limits of marginal dynamics for locally interacting processes on large sparse networks
  – provides an alternative to mean-field approximations
- Many interesting theoretical questions: to gain a better understanding of the local equations and looking at more general settings
- Interesting computational questions
- Variety of applications ...

Thank you !!