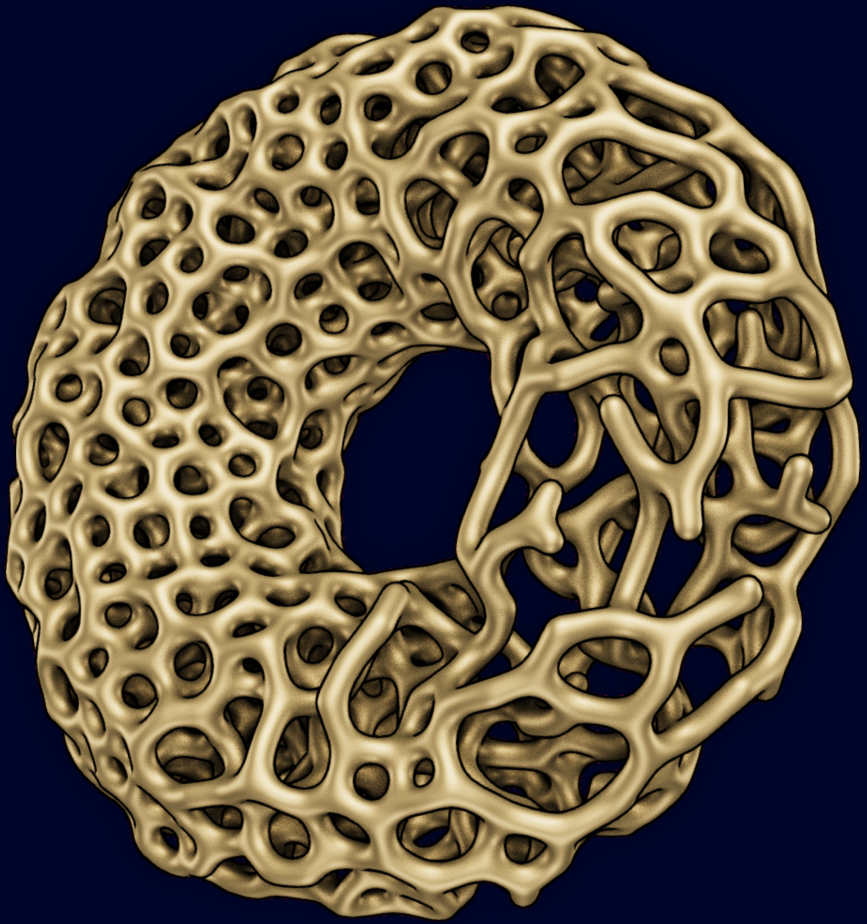

Interlacing Patterns in Shapes

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September 1, 2025. Nature sculpts fascinating patterns from minerals and biological tissues. We describe the "orderly randomness" of these morphologies as a *shape texture*. The curvatubes model considers shapes that minimize the curvature functional $\mathbf{F}(\mathcal{S})$ given by

$$\int_{\mathcal{S}} (a_{2,0}\kappa_1^2 + a_{1,1}\kappa_1\kappa_2 + a_{0,2}\kappa_2^2 + a_{1,0}\kappa_1 + a_{0,1}\kappa_2 + a_{0,0})\ dA$$

among smooth compact orientable surfaces $\mathcal{S}$ with principal curvatures κ_1 and κ_2. The coefficients $a_{i,j} \in \mathbb{R}$ encode the shape texture because different optimizers of the same functional share the same texture. The minimization is performed under a constraint of constant volume enclosed.

We approximate the geometric functional $\mathbf{F}$ by a phase-field energy of the form $\mathcal{F}_\epsilon \colon W^{2,2}(\Omega) \to \mathbb{R}$, whose mass-preserving flow

$$\dot{u} = \Delta \frac{\partial \mathcal{F}_\epsilon}{\partial u}(u)$$

is parameterized by a transition scale $\epsilon > 0$, a mass $m = \frac{1}{\Omega}\int_\Omega u \,\mathrm{d}x$ encoding the volume constraint, and the polynomial coefficients $\mathbf{a} \in \mathbb{R}^6$.

The phase-field u is initialized at random; it converges towards a local optimum u_∞ whose zero level set separates the two phases $+1$ and -1 and corresponds to the generated surface $\mathcal{S}$ [1]. The formulation can be altered to produce spatial heterogeneity by adopting space-varying parameters $\epsilon(x), m(x), \mathbf{a}(x)$. Domain constraints also allow us to confine the generation of the shape to a solid torus, for instance [2]. A well-chosen combination of these two ingredients leads to interlaced tubules that spread within the torus in a semi-controlled fashion.

[1] Song. Generation of Tubular and Membranous Shape Textures with Curvature Functionals. *J. Math. Imaging. Vis.* 2022.

[2] Song. The Geometry and Topology of Shape Patterns with Applications to Leukaemia. PhD Thesis, Imperial College London & The Francis Crick Institute. 2023.