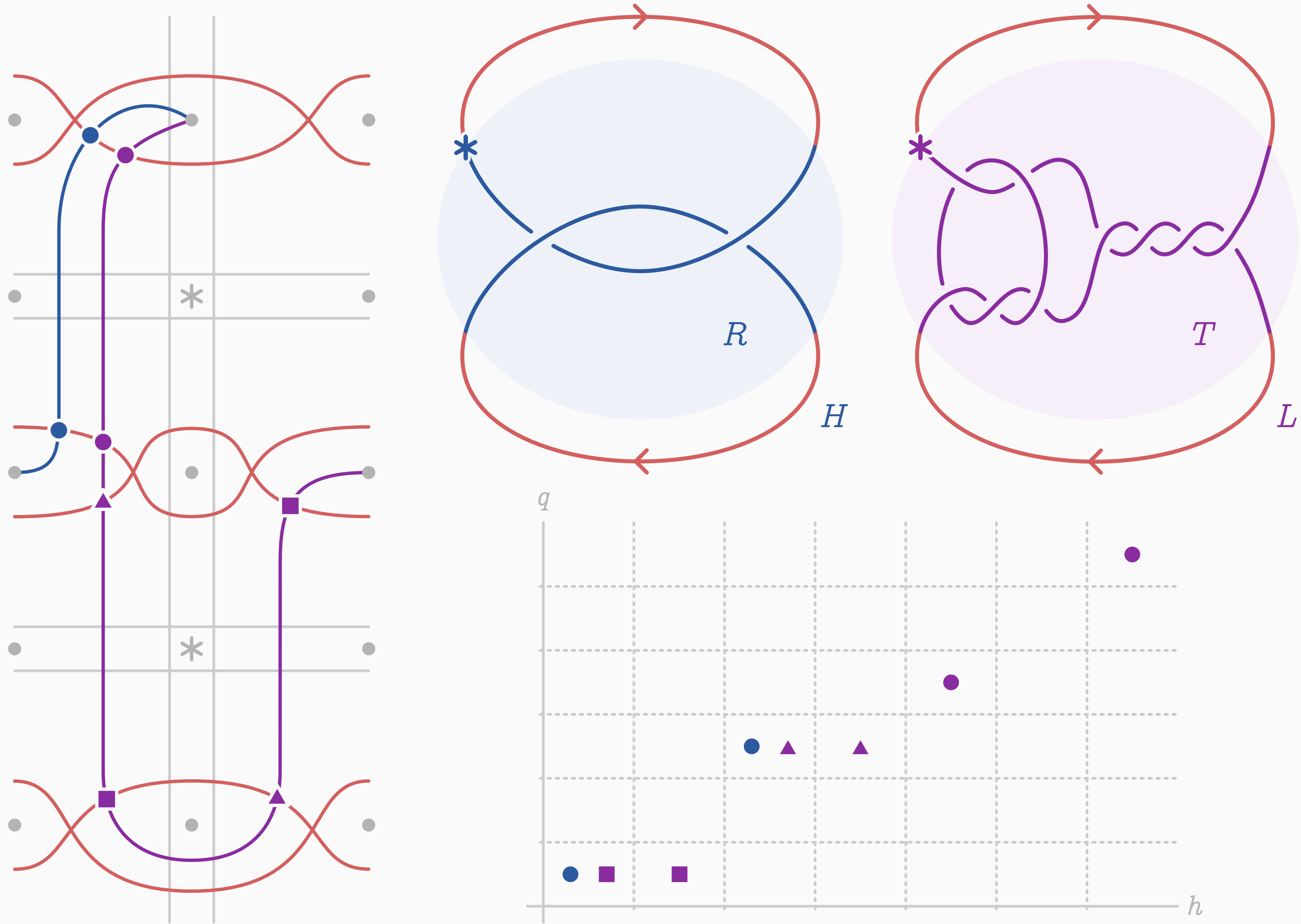
R
H
T
L
q
h

A Fake Hopf Link

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August 1, 2025. Examples of non-trivial links with trivial Jones polynomial are known to exist, at least for links with more than one component [3]. The question of whether the Jones polynomial detects the unknot U has been open since the polynomial's discovery in the early '80s [1]. Relaxing this question slightly, it is also unknown whether there exists a fake unknot, that is, a knot K with $V_K(t) = \pm t^n\, V_U(t)$ for some integer n.

The Hopf link H is the simplest non-trivial two-component link. On the obverse of this postcard, the reduced Khovanov homology

$$\widetilde{\mathrm{Kh}}(H) \cong \mathrm{HF}(m\widetilde{\mathrm{Kh}}({}^{*}\!\asymp), \widetilde{\mathrm{BN}}(R))$$

is computed via intersection of immersed curves associated with the link's constituent tangles [2]. Also computed is

$$\widetilde{\mathrm{Kh}}(L) \cong \mathrm{HF}(m\widetilde{\mathrm{Kh}}({}^{*}\!\asymp), \widetilde{\mathrm{BN}}(T))$$

where L is obtained from H by replacing a rational tangle R with a non-rational tangle T.

The Jones polynomial is recovered from the reduced Khovanov homology via a graded Euler characteristic:

$$V_L(t) = \textstyle\sum_{q,h} (-1)^h \dim \widetilde{\mathrm{Kh}}^q_h(L)\, t^{\frac{q}{2}}$$

It follows that $V_L(t) = t^3\, V_H(t)$, leading to:

Theorem. *There exists a fake Hopf link.*

[1] Jones. A Polynomial invariant of knots via von Neumann algebras. *Bull. Am. Math. Soc.* 1985.
[2] Kotelskiy, Watson & Zibrowius. Immersed curves in Khovanov homology. Preprint.
[3] Eliahou, Kauffman & Thistlethwaite. Infinite families of links with trivial Jones polynomial. *Topology.* 2003.