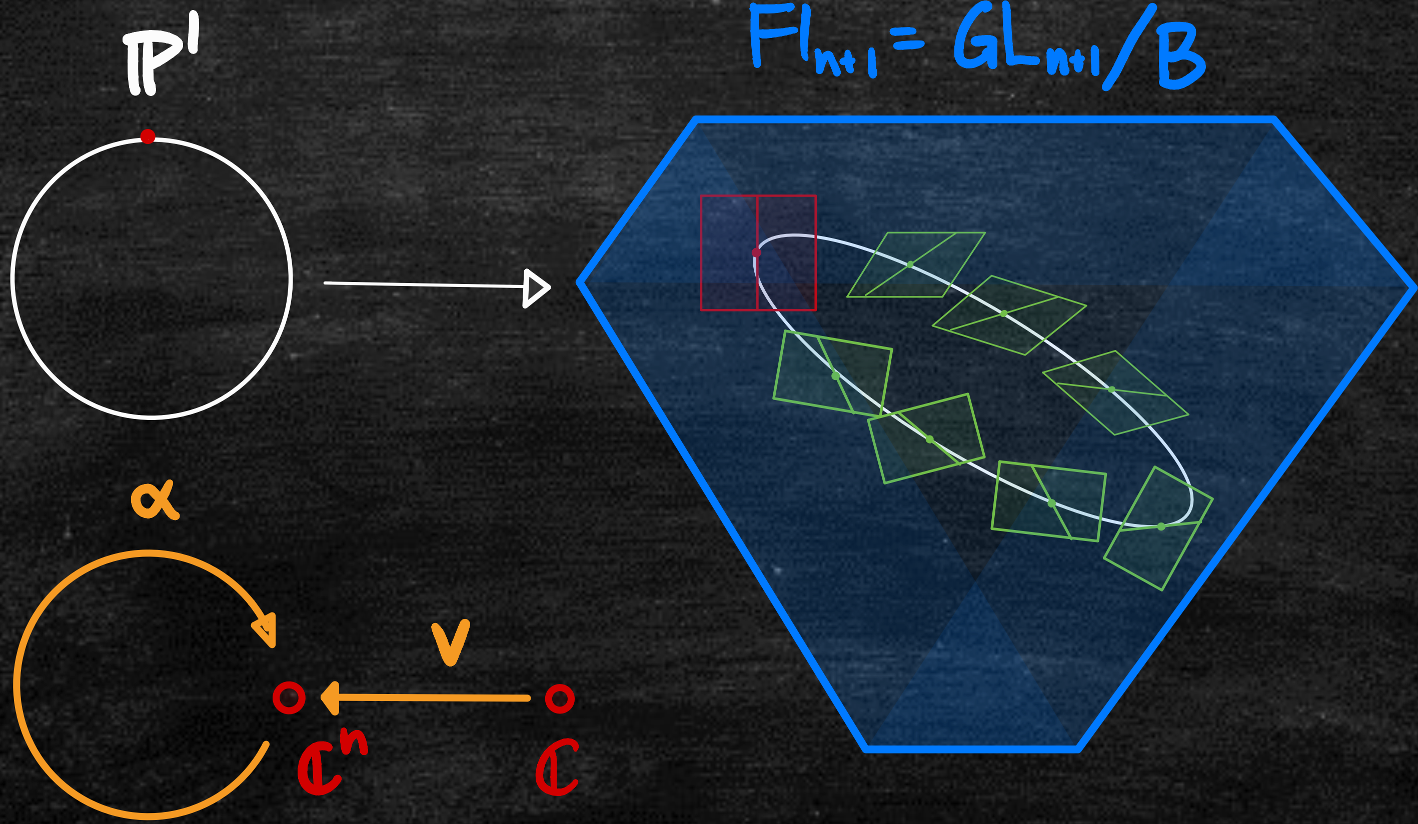

$\mathbb{P}^1$
$Fl_{n+1} = GL_{n+1}/B$
α
v
$\mathbb{C}^n$
$\mathbb{C}$

Based Linear Maps to the Flag Variety

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February 18, 2025. A map to the flag variety $\mathrm{Fl}_{n+1} = \mathrm{GL}_{n+1}/\mathrm{B}$ from $\mathbb{P}^1$ is *linear* if it is given by $(x_0 : x_1) \mapsto g_0 x_0 + g_1 x_1$, modulo elements of B, where $g_0, g_1 \in \mathfrak{gl}_{n+1}$; it is *based* if $(1:0) \mapsto \mathrm{Id}$, again modulo elements of B.

Theorem. *The space of based linear maps $\mathbb{P}^1 \to \mathrm{Fl}_{n+1}$ is isomorphic to $\mathrm{GL}_n \times \mathbb{C}^n$.*

Proof: For a based linear map, we may pick a representative in the mod B equivalence class where $g_0 = \mathrm{Id}$. Then the map is uniquely determined by the first n columns of g_1 , which in turn is determined by $(\alpha, v) \in \mathfrak{gl}_n \times \mathbb{C}^n$ as follows:

$$g_1 = \left(\begin{array}{c|c} & * \\ -\alpha^t & \vdots \\ & * \\ \hline v^t & * \end{array}\right)$$

The corresponding map to Fl_{n+1} is well defined if and only if the first n columns of $\mathrm{Id}\, x_0 + g_1 x_1$ are independent for all $\lambda = x_0/x_1$. This condition is equivalent to the following:

$$\begin{aligned} & \mathrm{Ker}(\alpha^t - \lambda\,\mathrm{Id}) \cap \mathrm{Ker}(v^t) = 0 \quad \forall \lambda \in \mathbb{C} \\ \iff & \mathrm{Im}(\alpha - \lambda\,\mathrm{Id}) \oplus \mathrm{Im}(v) = \mathbb{C}^n \quad \forall \lambda \text{ an eigenvalue of } \alpha \\ \iff & \nexists \text{ a degree } n-1 \text{ polynomial } p \text{ such that } p(\alpha)v = 0 \\ \iff & \{v, \alpha v, \ldots, \alpha^{n-1} v\} \text{ spans } \mathbb{C}^n \\ \iff & (\alpha, v) \in f^{-1}(\mathrm{GL}_n \times \mathbb{C}^n) \end{aligned}$$

In the last step $f : \mathfrak{gl}_n \times \mathbb{C}^n \to \mathfrak{gl}_n \times \mathbb{C}^n$ is the map $(\alpha, v) \mapsto (\beta, w)$ where $\beta = (v|\alpha v| \ldots |\alpha^{n-1} v)$ is the matrix whose k^{th} column is the vector $\alpha^{k-1} v$ and $w = \alpha^n v$. The theorem then follows from the observation that $f|_{f^{-1}(\mathrm{GL}_n \times \mathbb{C}^n)}$ is an isomorphism onto its image with inverse given by

$$\alpha = \beta \left(\begin{array}{cccc|c} 0 & & & & \vdots \\ 1 & \ddots & & & \beta^{-1} w \\ & \ddots & 0 & & \vdots \\ & & 1 & & \vdots \end{array}\right) \beta^{-1}$$

and v is given by the 1^{st} column of β ■

The open set $f^{-1}(\mathrm{GL}_n \times \mathbb{C}^n)$ is given by the condition that (α, v) is a stable representation of the quiver shown on the front of this card.