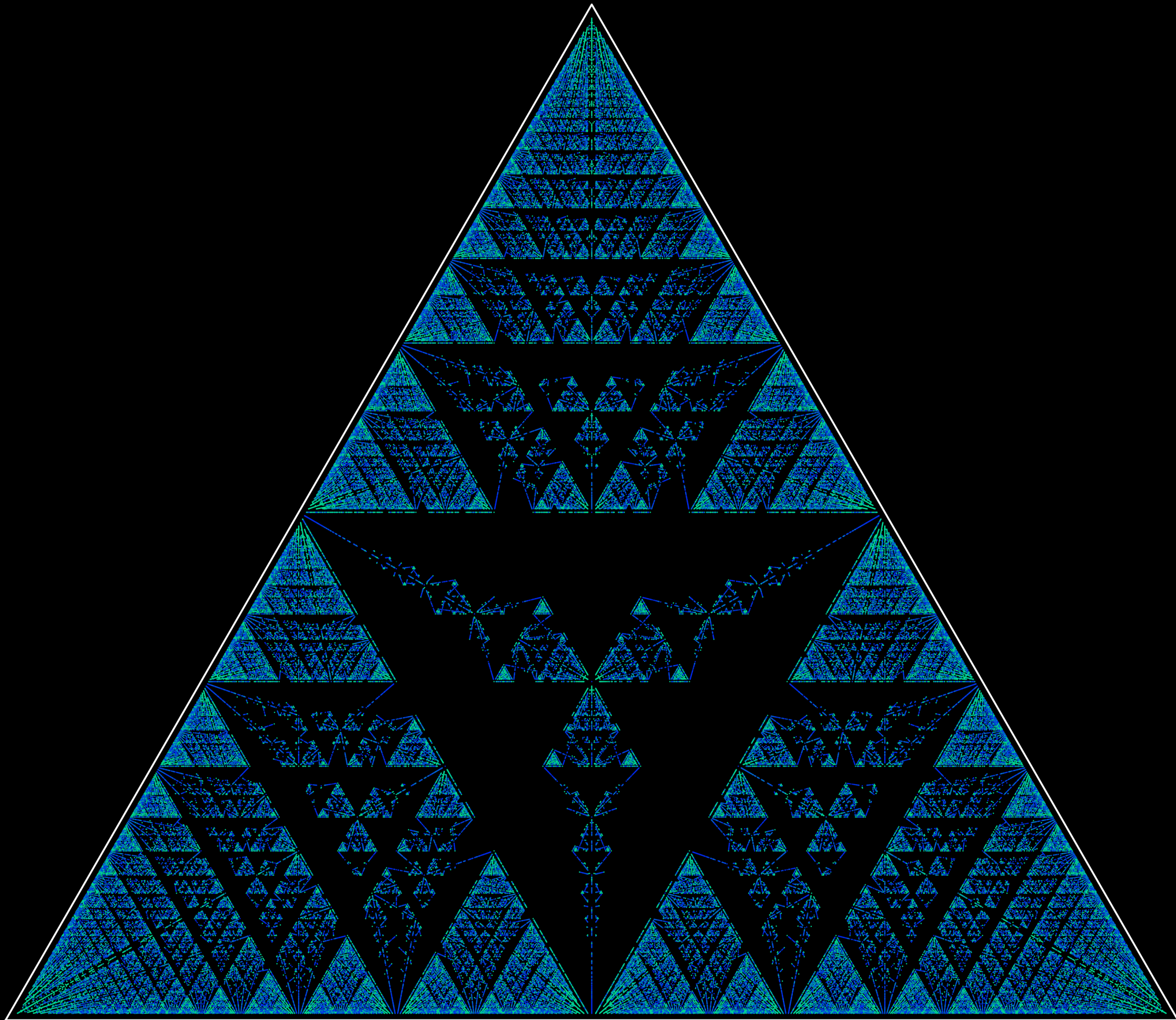

The Small Seifert Fibered Embeddahedron

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August 10, 2022. Given any oriented 3-manifold M, it is well known that there exists a smooth oriented 4-manifold W with boundary M. What if we further require the 4-manifold W to have negative semi-definite intersection pairing $H_2(W) \times H_2(W) \to \mathbb{Z}$? Not all 3-manifolds M bound such a 4-manifold, and understanding precisely which ones do turns out to be a surprisingly subtle problem that is related to the fractal on this postcard. Answering this question for families of 3-manifolds with various additional conditions imposed on W has led to many important and beautiful results concerning knots and 3-manifolds.

For example, Lisca characterised precisely which 2-bridge knots bound a smooth disk in the 4-ball [1]. He did so by answering this question when M is a lens space and $b_2(W) = 0$. Greene determined the set of lens spaces arising as integral surgery on a knot by answering this question when M is a lens space space and W is the trace of a knot surgery [2].

Now consider this question when the 3-manifold M is a small Seifert fibered space oriented to bound a positive semi-definite plumbing 4-manifold. We can parameterize any such 3-manifold as $M = S^2(e; \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3})$ where $e \in \mathbb{N}$ is the central weight and $0 < r_1, r_2, r_3 < 1$ are rational numbers with $r_1 + r_2 + r_3 \leq e$; see [3] for notation. Work in [3] analyzes the case $e > 1$, leaving $e = 1$ as the least understood case. For $e = 1$, Y is parameterized by points (r_1, r_2, r_3), which form a rational tetrahedron $\mathcal{T}$ in $\mathbb{Q}^3$.

Let $\mathcal{T}_\partial \subseteq \mathcal{T}$ be the points for which M bounds some smooth negative semi-definite 4-manifold. Donaldson's diagonalization theorem gives a combinatorial lattice-based obstruction to this problem, and we denote by $\mathcal{T}_\partial^d \subseteq \mathcal{T}$ the points that pass this obstruction, so that by definition $\mathcal{T}_\partial \subseteq \mathcal{T}_\partial^d$.

What does the subset $\mathcal{T}_\partial^d$ of the tetrahedron look like? This postcard shows the face of the tetrahedron $\mathcal{T}$ corresponding to $r_1 + r_2 + r_3 = 1$ with points in $\mathcal{T}_\partial^d$ colored in blue/green. The different shades of blue/green correspond to different minimal codimensions of lattice maps passing the obstruction. The mesmerizing picture that results hints at the perplexing fractal-like nature of this subtle problem.

[1] Lisca. Lens spaces, rational balls and the ribbon conjecture. *Geom. Topol.*, 2007.
[2] Greene. The lens space realization problem. *Ann. of Math.*, 2013.
[3] Issa & McCoy. On Seifert fibered spaces bounding definite manifolds. *Pacific J. Math.*, 2020.

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