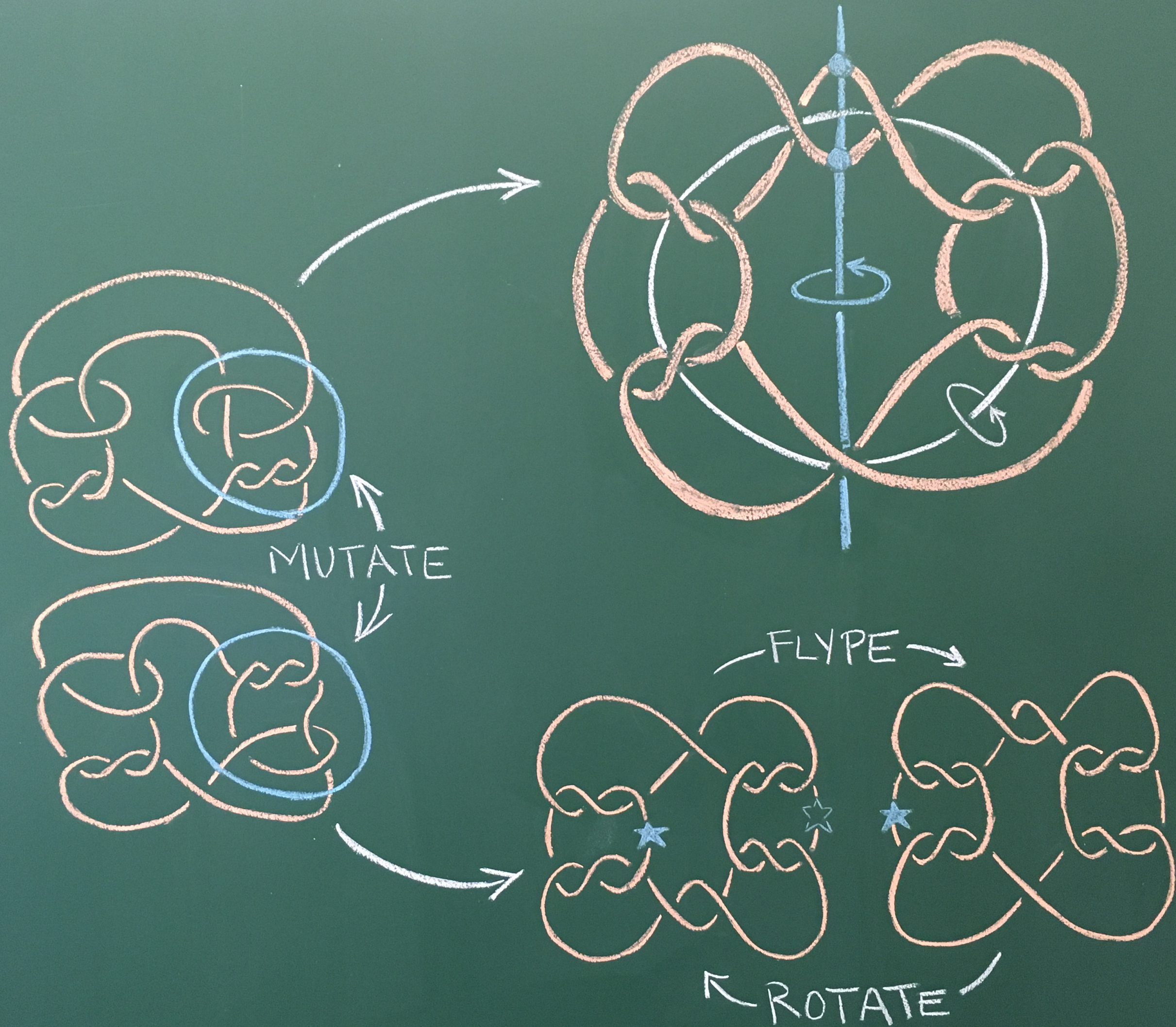
MUTATE
FLYPE
ROTATE

Symmetry & Mutation

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April 28, 2021. When Conway set about tabulating knots in the 1960s, appealing to a divide-and-conquer method involving tangles, he came up against an interesting ambiguity when certain tangles are reassembled [2]. This became known as mutation—the result of removing a tangle, rotating it about an axis, and replacing it—and while it can alter the given knot, detecting the difference can be extremely subtle. Indeed, the original example led to a pair of knots later separated by Riley, appealing to representations of the knot group [5].

In general, mutation is invisible to a long list of knot invariants, some of which are remarkably powerful; on the other hand the question *what invariants see mutation?* is a tricky one. At least among prime knots, owing to a celebrated result of Gordon and Luecke [3], the knot group will always separate distinct mutants, but this goes from one hard problem to another; deciding if two groups are different is not easy. While anachronistic, Riley made use of this strength of the knot group by appealing to a robust invariant of groups in order to separate Conway's mutants.

Another example is shown on the front of the card. In this case a refinement of Khovanov homology taking a strong inversion into account is able to establish that the mutants are distinct [4]. A strong inversion is a type of involution and can be viewed as an element of the symmetry group of a knot; see [6]. This latter is the group of diffeomorphisms of the knot complement, up to isotopy. In this case, it turns out that the symmetry group itself separates the two mutants: one has symmetry group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (a 4-element group) while the the other has dihedral symmetry group D_4 (an 8-element group).

For the smaller of the two groups, a generating set has been shown on the front of the card. For the larger group, the order 4 element is obtained by composing a flype with a rotation; see Boyle [1].

In the end a principle similar to that applied by Riley is in play: The symmetry group of a hyperbolic knot agrees with the outer-automorphism group of the knot group. On the other hand, I was surprised to see this work, in practice, on such a simple example. And it seems worth recording:

Theorem. *Mutants that are separated by the symmetry group exist.*

Moreover, this example shows that the symmetry group does the job even in cases where neither knot Floer homology nor Khovanov homology is able to make the distinction.

[1] Boyle. Involutions of alternating links. *Proc. Amer. Math. Soc.*, to appear.
[2] Conway. An enumeration of knots and links. In *Computational Problems in Abstract Algebra*, 1970.
[3] Gordon & Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 1989.
[4] Lobb & Watson. A refinement of Khovanov homology. *Geom. Topol.*, to appear.
[5] Riley. Homomorphisms of knot groups on finite groups. *Math. Comp.*, 1971.
[6] Sakuma. On strongly invertible knots. In *Algebraic and topological theories*, 1986.