PUTNAM PRACTICE SET 9

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Problem 1. Let S be the set of real numbers which is closed under multiplication, i.e., if $a, b \in S$ then $ab \in S$. Let T and U be disjoint subsets of S whose union is S. Given that the product of 3 elements of T (not necessarily distinct) is also contained in T, and similarly, the product of 3 elements of U is also contained in U, the prove that at least one of the two sets T or U is closed under multiplication.

Solution. Suppose there exist $t_1, t_2 \in T$ with $t_1t_2 \in U$ and also, there exist $u_1, u_2 \in U$ such that $u_1u_2 \in T$. Then

$$t_1t_2u_1u_2 = t_1 \cdot t_2 \cdot (u_1u_2) \in T$$

but also

$$t_1t_2u_1u_2 = u_1 \cdot u_2 \cdot (t_1t_2) \in U,$$

contradiction. So, at least one of the two subsets T or U must be closed under multiplication.

Problem 2. Let $x_1(t), \dots, x_n(t)$ be differentiable functions satisfying the following system of differential equations:

$$x_i'(t) = \sum_{j=1}^{n} a_{i,j}x_j(t),$$

for given positive real numbers $a_{i,j}$. If

$$\lim_{t\to\infty} x_i(t) = 0 \text{ for each } i = 1, \dots, n,$$

then prove that the functions $x_1(t), \dots, x_n(t)$ are linearly dependent, i.e., there exist constants $c_1, \dots, c_n$ (not all equal to 0) such that

$$\sum_{i=1}^{n} c_ix_i(t) = 0.$$

Solution. The vector solutions $\vec{x}(t)$ of a linear system of differential equations is of the form $\sum_{i=1}^{n} b_if_i(t) \cdot \vec{v_i}$, where the vectors $\vec{v_i}$ are linearly independent, the b_i's are constants and the $f_i(t)$ are functions. Furthermore, if λ_i is an eigenvalue for the corresponding matrix $A = (a_{i,j})_{1\le i,j\le n}$, then we may take $f_i(t) = e^{\mathrm{Re}(\lambda_i)\cdot t}$ (where $\mathrm{Re}(z)$ is always the real part of the complex number z).

Now, since each $a_{i,j}$ is a positive real number, then the trace of A is strictly positive and therefore, there is at least one eigenvalue λ_i whose real part is strictly positive. Then there exists a function $f_i(t)$ which does not converge to 0 as $t \to \infty$. But then, let $\vec{w}$ be a nonzero vector orthogonal to all vectors $\vec{v_j}$ for $j \neq i$. We have that

$$\vec{w} \cdot \vec{x}(t) = b_if_i(t)(\vec{w} \cdot \vec{v_i}) + \sum_{j\neq i} b_jf_j(t)(\vec{w} \cdot \vec{v_j}) = b_if_i(t) \cdot d_0,$$

where $d_0 := \vec{w} \cdot \vec{v_i} \neq 0$. On the other hand, since each $x_i(t) \to 0$ as $t \to \infty$, then also $\vec{w} \cdot \vec{x}(t) \to 0$ as $t \to \infty$ and thus, according to the above computation, coupled with the fact that $f_i(t)$ does not converge to 0 as $t \to \infty$ (it actually diverges to $+\infty$ in this case), then we must have that $b_i = 0$. But then we conclude that $\vec{w} \cdot \vec{x}(t) = 0$, i.e., $x_1(t), \cdots, x_n(t)$ are linearly dependent, as claimed.

Problem 3. Let p be a prime number greater than 3 and let $k = \left[\frac{2p}{3}\right]$, where $[z]$ denotes (as always) the integer part of the real number z (i.e., the largest integer less than or equal to z). Prove that p^2 divides $\sum_{i=1}^{k} \binom{p}{i}$.

Solution. We have that for each $1 \leq i \leq k$ that

$$\binom{p}{i} = p \cdot \frac{(p-1)\cdots(p-i+1)}{i!}$$

and since p is a prime number not dividing $i!$ and furthermore, $\binom{p}{i}$ is an integer, then we must have that

$$\frac{(p-1)\cdots(p-i+1)}{i!} \text{ is an integer.}$$

Now, clearly, $(p-1)\cdots(p-i+1) = p\ell_i + (-1)^{i-1}(i-1)!$ for some integer ℓ_i because

$$(p-1)\cdots(p-i+1) \equiv (-1)\cdots(-i+1) \equiv (-1)^{i-1} \cdot (i-1)! \pmod{p}.$$

Therefore, there exists some integer b_i such that letting a_i be an integer with the property that $a_i \cdot i \equiv 1 \pmod{p}$, we have that

$$\frac{(p-1)\cdots(p-i+1)}{i!} = pb_i + (-1)^{i-1} \cdot a_i.$$

So, we are left to prove that p must divide $\sum_{i=1}^{k}(-1)^{i-1}a_i$. We split our analysis into two cases:

Case 1. $p = 6c+1$ for an integer c, in which case, $k = 4c$. Then

$$\sum_{i=1}^{4c}(-1)^{i-1}a_i = \sum_{i=1}^{4c} a_i - \sum_{i=1}^{2c} 2a_{2i}$$

and since $a_{2i} \cdot (2i) \equiv 1 \pmod{p}$, while $a_i \cdot i \equiv 1 \pmod{p}$, we must have that $2a_{2i} - a_i \equiv 0 \pmod{p}$ for each $1 \leq i \leq 2c$ and so,

$$\sum_{i=1}^{4c}(-1)^{i-1}a_i \equiv \sum_{i=2c+1}^{4c} a_i \pmod{p}.$$

However, for each $3c+1 \leq i \leq 4c$, we have that $a_i \equiv -a_{6c+1-i} \pmod{p}$ (note that $p = 6c+1$). So,

$$\sum_{i=1}^{4c}(-1)^{i-1}a_i \equiv \sum_{i=2c+1}^{3c} a_i - \sum_{i=2c+1}^{3c} a_i \equiv 0 \pmod{p},$$

as desired.

Case 2. $p = 6c - 1$ and so, $k = 4c - 1$. Then (arguing as before)

$$\begin{aligned}
&\sum_{i=1}^{4c-1} (-1)^{i-1} a_i \\
&\qquad \equiv \sum_{i=1}^{4c-1} a_i - \sum_{i=1}^{2c-1} 2a_{2i} \pmod p \\
&\qquad \equiv \sum_{i=1}^{4c-1} a_i - \sum_{i=1}^{2c-1} a_i \pmod p \\
&\qquad \equiv \sum_{i=2c}^{4c-1} a_i \pmod p \\
&\qquad \equiv \sum_{i=2c}^{3c-1} a_i + \sum_{i=3c}^{4c-1} a_i \pmod p \\
&\qquad \equiv \sum_{i=2c}^{3c-1} a_i - \sum_{i=3c}^{4c-1} a_{6c-1-i} \pmod p \\
&\qquad \equiv \sum_{i=2c}^{3c-1} a_i - \sum_{j=2c}^{3c-1} a_j \pmod p \\
&\qquad \equiv 0 \pmod p,
\end{aligned}$$

as desired.

Problem 4. Let c be a positive real number. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the property that for each real number x, we have that $f(x) = f(x^2 + c)$.

Solution. First we note that $f(-x) = f(x^2 + c) = f(x)$ and so, f must be an even function; so, it suffices to describe $f(x)$ for $x \in [0, +\infty)$ and then simply define $f(-x) = f(x)$ for $x > 0$.

Now, there are two cases:

Case 1. $0 < c \leq \frac{1}{4}$.

In this case, there are real roots for the equation $x^2 + c - x = 0$; we denote them (in increasing order) by r_1 and r_2 and we note that it could be that $r_1 = r_2$ (if $c = \frac{1}{4}$). Also, we note that $r_1 > 0$ because $c > 0$ (and so, also $r_2 > 0$). We split our analysis on each of the three intervals $(0, r_1)$, (r_1, r_2) and $(r_2, +\infty)$ (with the observation that the middle interval would not exist if $c = \frac{1}{4}$).

Case 1a. For $x \in (0, r_1)$ we consider the sequence $\{x_n\}$ defined by $x_0 = x$ and then recursively as $x_{n+1} = x_n^2 + c$. Since $x_0 < r_1$, we have that

$$x_1 = x_0^2 + c > x_0$$

but also

$$x_1 = x_0^2 + c < r_1^2 + c = r_1.$$

So, $0 < x_0 < x_1 < r_1$ and a simple inductive argument yields that the sequence $\{x_n\}$ is strictly increasing, contained inside the interval $(0, r_1)$. So, its limit must be r_1 because $r_1^2 + c = r_1$. Therefore, using the continuity of $f(x)$, we get that on the interval $(0, r_1)$, we have that

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = f(r_1),$$

i.e., $f(x)$ is constant on $(0, r_1)$.

Case 1b. For $x \in (r_1, r_2)$ (which automatically means that $0 < c < \frac{1}{4}$), again considering the sequence $\{x_n\}$ defined recursively as above starting with $x_0 = x$, we observe that

$$x_1 = x_0^2 + c < x_0$$

but also

$$x_1 = x_0^2 + c > r_1^2 + c = r_1$$

and so, inductively, we have that the sequence $\{x_n\}$ decreases inside the interval (r_1, r_2) and its limit is r_1. Therefore, arguing as before (using the continuity of $f(x)$), we must have that $f(x)$ is constant on (r_1, r_2).

Case 1c. For $x \in (r_2, +\infty)$, the previously defined sequence $\{x_n\}$ diverges to $+\infty$, so it is no longer useful. However, we may define a new sequence $\{y_n\}$ starting with $y_0 = x$ and then $y_{n+1} = \sqrt{y_n - c}$. Then

$$y_1 = \sqrt{y_0 - c} > \sqrt{r_2 - c} = r_2$$

but more importantly,

$$y_1 = \sqrt{y_0 - c} < y_0,$$

which means that an inductive argument yields that $\{y_n\}$ decreases and its limit is r_2. So, once again on the interval $(r_2, +\infty)$, we obtain that $f(x)$ must be constant (due to its continuity).

Finally, putting together all our findings from Cases 1a, 1b, 1c (along with the fact that f is an even function), we conclude that if $0 < c \le \frac{1}{4}$, then $f(x)$ must be a constant function.

Case 2. $c > \frac{1}{4}$.

We consider now the sequence $\{z_n\}$ given by $z_0 = 0$ and recursively $z_{n+1} = z_n^2 + c$. Clearly, $z_{n+1} > z_n$ for all n and moreover, the sequence diverges to $+\infty$. Now, we see that it suffices to choose any continuous function on the interval $[0, c] = [z_0, z_1]$ with the property that $f(0) = f(c)$ and then define recursively $f(x^2 + c) = f(x)$ which would allow us to define $f(x)$ on intervals $[z_1, z_2]$ and inductively we define $f(x)$ on each interval $[z_n, z_{n+1}]$. Then we also extend the definition of $f(x)$ for negative real numbers using the fact that f is an even function. So, in this case, there are a continuum of desired functions $f(x)$; they're all uniquely determined by a choice of a continuous function on $[0, c]$ with the only restriction that $f(0) = f(c)$.