PUTNAM PRACTICE SET 4

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Problem 1. Let $\{F_n\}_{n\ge 1}$ be the Fibonacci sequence, i.e.,

$$F_1 = 1,\ F_2 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for each } n \ge 1.$$

Find all positive real numbers a and b with the property that for each $n \ge 1$, we have that $aF_n + bF_{n+1}$ is another element of the Fibonacci sequence.

Solution. We let $\alpha : \mathbb{N} \longrightarrow \mathbb{N}$ be the function having the property that

$$aF_n + bF_{n+1} = F_{\alpha(n)} \text{ for all } n \ge 1.$$

Adding the above identity for two consecutive integers, we get

$$\begin{aligned} &F_{\alpha(n+1)} + F_{\alpha(n)} \\ &\qquad = (aF_{n+1} + bF_{n+2}) + (aF_n + bF_{n+1}) \\ &\qquad = a(F_{n+1} + F_n) + b(F_{n+2} + F_{n+1}) \\ &\qquad = aF_{n+2} + bF_{n+3} \\ &\qquad = F_{\alpha(n+2)}. \end{aligned}$$

Now, for any $r, s, t \ge 3$ we have that $F_r + F_s = F_t$ if and only if $r + 2 = s + 1 = t$ (after possibly interchanging r and s). Indeed, in order to prove this, we note that we may assume without loss of generality that $r \le s$ and also, we definitely need $t \ge s + 1$. Furthermore, for $3 \le r \le s$, we observe that we cannot have $r = s$ since

$$F_r + F_s = 2F_s < F_s + F_{s+1} = F_{s+2} \text{ and also } F_r + F_s > F_{s-1} + F_s = F_{s+1}.$$

So, indeed, $r < s < t$. Now, if $t \ge s + 2$, then

$$F_t \ge F_{s+2} = F_{s+1} + F_s > F_s + F_r.$$

So, we must have that $s+1 = t$ and then $F_t - F_s = F_{s-1}$, which means that $r = s-1$ (note that $r \ge 3$ and so, $F_r > 1$). Therefore, $F_r + F_s = F_t$ with $3 \le r \le s \le t$ occurs only if $r + 2 = s + 1 = t$ yields that for $n \ge 2$ (which would guarantee that $\alpha(2) > 1$) we have that

$$F_{\alpha(n)} + F_{\alpha(n+1)} = F_{\alpha(n+2)},$$

which yields $\alpha(n) + 2 = \alpha(n+1) + 1 = \alpha(n+2)$. So, $\alpha(n) = n + c$ for some integer c. On the other hand, a simple induction - using the general form of the Fibonacci sequence:

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right),$$

we obtain that

$$F_{c-1}F_n + F_cF_{n+1} = F_{n+c}.$$

So, for any (a, b) of the form (F_{c-1}, F_c) for $c \ge 2$ (so that both a and b need to be strictly positive) we obtain the desired conclusion for this *Problem.*

Problem 2. For any polynomial $P \in \mathbb{C}[x]$ and for each complex number a, we denote by P_a the set of all $z_0 \in \mathbb{C}$ such that $P(z_0) = a$. Let $P, Q \in \mathbb{C}[x]$ such that $P_2 = Q_2$ and $P_5 = Q_5$. Prove that $P = Q$.

Solution. Let $\alpha_1, \dots, \alpha_r$ be the elements of $P_2 = Q_2$ and also, let $\beta_1, \dots, \beta_s$ be the elements of $P - 5 = Q_5$; then we let $k_1, \dots, k_r$ be the respective multiplicities of $\alpha_1, \dots, \alpha_r$ as roots for $P(x) - 2$, and also we let $m_1, \dots, m_s$ be the respective multiplicities of $\beta_1, \dots, \beta_s$ as roots of $P(x) - 5$. Then $k_1 - 1, \dots, k_r - 1$ are the multiplicities of $\alpha_1, \dots, \alpha_r$ as roots of $P'(x)$ and similarly, $m_1 - 1, \dots, m_s - 1$ are the multiplicities of $\beta_1, \dots, \beta_s$ as roots of $P'(x) = 0$. Then - letting d be the degree of $P(x)$ - we get that

$$d - 1 \geq \sum_{i=1}^{r} (\alpha_i - 1) + \sum_{j=1}^{s} (\beta_j - 1).$$

On the other hand, we have $d = \sum_{i=1}^{r} \alpha_i$ and $d = \sum_{j=1}^{s} \beta_j$ (since these are the multiplicities of the roots of $P(x) - 2$, respectively of $P(x) - 5$); so, we obtain

$$d - 1 \geq 2d - r - s, \text{ which yields } r + s \geq d + 1.$$

Similarly, we get $r + s \geq D + 1$, where D is the degree of $Q(x)$. Thus we get that the polynomial $P(x) - Q(x)$, which has degree bounded above by $\max\{d, D\}$ has at least $r + s$ roots $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$; in conclusion, we must have $P = Q$ identically.

Problem 3. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We have a function $f : \mathbb{N}_0^2 \longrightarrow \mathbb{N}_0$ satisfying the following properties:

- $f(0, y) = y + 1$ for each $y \in \mathbb{N}_0$;
- $f(x + 1, 0) = f(x, 1)$ for each $x \in \mathbb{N}_0$; and
- $f(x + 1, y + 1) = f(x, f(x + 1, y))$ for each $x, y \in \mathbb{N}_0$.

Find $f(4, 2019)$.

Solution. We have $f(1, y + 1) = f(0, f(1, y)) = f(1, y) + 1$ for all $y \in \mathbb{N}_0$ and so, $f(1, y) = f(1, 0) + y = f(0, 1) + y = 2 + y$. Next, we find

$$f(2, y + 1) = f(1, f(2, y)) = f(2, y) + 2 \text{ for all } y \in \mathbb{N}_0$$

and so, $f(2, y) = f(2, 0) + 2y = f(1, 1) + 2y = 3 + 2y$ for all $y \in \mathbb{N}_0$. Next,

$$f(3, y + 1) = f(2, f(3, y)) = 3 + 2f(3, y).$$

Thus, letting $g(y) := f(3, y) + 3$, we obtain that

$$g(y + 1) = 2g(y) \text{ for all } y \in \mathbb{N}_0$$

and so, $f(3, y) = 2^y(f(3, 0) + 3) - 3 = 2^y(f(2, 1) + 3) - 3 = 2^y \cdot 8 - 3 = 2^{y+3} - 3$. Finally,

$$f(4, y + 1) = f(3, f(4, y)) = 2^{f(4,y)+3} - 3$$

for all $y \in \mathbb{N}_0$. We let $h(y) := f(4, y) + 3$ and thus

$$h(y + 1) = 2^{h(y)} \text{ for all } y \in \mathbb{N}_0$$

and so, $h(y) = 2^{2^{2^{\cdots}}}$, where there are $y + 3$ iterated powers of 2 (note that $h(0) = f(4, 0) + 3 = f(3, 1) + 3 = 2^4 = 2^{2^2}$). Hence

$$f(4, 2019) = 2^{2^{2^{2^{\cdots}}}} - 3,$$

where there are 2022 iterated powers of 2 in the above formula.

Problem 4. We consider all possible sequences $\{x_n\}_{n\geq 0}$ of positive real numbers having the properties that $x_0 = 1$ and also that $x_{n+1} \leq x_n$ for each $n \geq 0$.

(I) Prove that for each such sequence $\{x_n\}_{n\geq 0}$, we have that the series
$$\sum_{i=0}^{\infty} \frac{x_i^2}{x_{i+1}}$$
is either divergent to $+\infty$, or it coverges to a real number at least equal to 4.

(II) Prove that there exists exactly one such sequence $\{x_n\}_{n\geq 0}$ for which the series
$$\sum_{i=0}^{\infty} \frac{x_i^2}{x_{i+1}}$$
equals 4.

Solution. We let $\{y_n\}_{n\geq 1}$ with the property that
$$x_n = \prod_{i=1}^{n} y_i \text{ for each } n \geq 1;$$
in other words, $y_n := \frac{x_n}{x_{n-1}}$. By our hypothesis, we have that $y_n \leq 1$ for each $n \geq 1$. So, the above series $\sum_{n\geq 0} \frac{x_n^2}{x_{n+1}}$ equals
$$\frac{1}{y_1} + \frac{y_1}{y_2} + \frac{y_1 y_2}{y_3} + \cdots + \frac{y_1 y_2 \cdots y_n}{y_{n+1}} + \cdots$$
We denote by $f(y_1, \dots, y_n, \dots)$ the above sum. Clearly,
$$f(y_1, \dots, y_n, \dots) = \frac{1}{y_1} + y_1 \cdot f(y_2, \dots, y_n, \dots).$$
So, letting f_0 be the smallest possible sum attained by a choice of some sequence $\{y_n\}$ and then we see that
$$f_0 = \frac{1}{y_1} + y_1 f_0.$$
So, $f_0 := \frac{1}{y_1(1-y_1)}$ and clearly, we see that the minimum value of f_0 is obtained for $y_1 = \frac{1}{2}$. So, the smallest value for the above sum of the series is 4 and it is attained when $y_1 = \frac{1}{2}$ and inductively, when each $y_n = \frac{1}{2}$, which translates to $x_n = \frac{1}{2^n}$.