PUTNAM PRACTICE SET 3

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Problem 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying the relation:

$$f(x + y + xy) = f(x) + f(y) + f(xy) \text{ for each } x, y, \in \mathbb{R}.$$

Prove that $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

Solution. Letting $x = y = 0$ we obtain $f(0) = 3f(0)$ and so, $f(0) = 0$. Then letting $y = -1$ (and x arbitrary) we obtain

$$f(-1) = f(x) + f(-1) + f(-x),$$

which yields $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Now, we simply replace x and y by $-x$, respectively $-y$ and obtain

$$f(xy - x - y) = f(xy) + f(-x) + f(-y) = f(xy) - f(x) - f(y)$$

which combined with the main relation yields

$$f(xy - (x + y)) + f(xy + (x + y)) = 2f(xy).$$

Now, for fixed $xy =: a$, we observe that $x+y$ varies on the entire set of real numbers (i.e., it can be arbitrarily large and negative and also arbitrarily large and positive). This proves that for all $a, b \in \mathbb{R}$ we have

$$f(a - b) + f(a + b) = 2f(a).$$

However, letting $a = b$ in the above expression we get that

$$f(0) + f(2a) = 2f(a) \text{ and so, } f(2a) = 2f(a) \text{ because } f(0) = 0.$$

Thus, $f(a - b) + f(a + b) = f(2a)$ for all $a, b \in \mathbb{R}$ which yields the relation asked in the problem.

Problem 2. Find all positive real numbers a with the property that the equation $\log_a(x) - x = 0$ has exactly one real solution.

Solution. We split our analysis into several cases:

Case 1. $0 < a < 1$.

In this case, $\log_a(x)$ decreases from $+\infty$ to $-\infty$, while x increases from 0 to $+\infty$; so, using that $f(x) := \log_a(x) - x$ is a continuous function (on $(0, +\infty)$), then we conclude that for each $a \in (0, 1)$ there exists a unique $x \in (0, +\infty)$ such that $f(x) = 0$, i.e., $\log_a(x) = x$.

Case 2. $a > 1$.

In this case the derivative of the above defined function $f(x)$ is

$$f'(x) = \frac{1}{x \cdot \ln(a)} - 1$$

and so, $f(x)$ is increasing on $(0, 1/\ln(a))$, while $f(x)$ is decreasing on $(1/\ln(a), +\infty)$. We compute the global maximum of $f(x)$ on $(0, +\infty)$:

$$f\left(\frac{1}{\ln(a)}\right) = \frac{\ln\left(\frac{1}{\ln(a)}\right)}{\ln(a)} - \frac{1}{\ln(a)} = -\frac{\ln(\ln(a)) + 1}{\ln(a)}.$$

Now, if the global maximum of $f(x)$ is 0 then there exists indeed a single value of x for which $\log_a(x) = x$; so,

Subcase 2(i). If $a = e^{\frac{1}{e}}$ then there exists a unique value of x such that $\log_a(x) = x$.

Now, if $\ln(\ln(a)) + 1 > 0$, then the global maximum of $f(x)$ is negative and therefore,

Subcase 2(ii). If $a > e^{\frac{1}{e}}$ then there exists no x such that $\log_a(x) = x$.

Finally, if $\ln(\ln(a)) + 1 < 0$, then the global maximum of $f(x)$ is positive and then we conclude that

Subcase 2(iii). If $1 < a < e^{\frac{1}{e}}$ then there exist exactly two values of x (one in the interval $(0, 1/\ln(a))$ and the other value in $(1/\ln(a), +\infty)$ since $\lim_{x\to 0^+} f(x) = \lim_{x\to+\infty} f(x) = -\infty$) such that $\log_a(x) = x$.

Problem 3.

(a) Find all integers $n > 2$ for which there exists an integer $m \geq n$ such that m divides the least common multiple of $m-1, m-2, \cdots, m-n+1$.

(b) Find all positive integers $n > 2$ for which there exists exactly one integer $m \geq n$ such that m divides the least common multiple of $m-1, m-2, \cdots, m-n+1$.

Solution. Let p^α be a prime power appearing in the prime power factorization of m. Then m dividing $\text{lcm}[m-1, \cdots, m-(n-1)]$ yields that p^α must divide one of the numbers $m-i$ (for $i = 1, \ldots, n-1$) and so, p^α must divide $m-(m-i) = i$. In conclusion, m divides $\text{lcm}[m-1, \cdots, m-(n-1)]$ if and only if m divides $\text{lcm}[1, \ldots, n-1] := L(n)$. So, the existence of at least one integer $m \geq n$ with the property that it divides $\text{lcm}[m-1, \cdots, m-(n-1)]$ is equivalent with asking that $L(n) \geq n$. Now, since $L(n) \geq (n-1)(n-2)$ and

$$(n-1)(n-2) \geq n \text{ for all } n \geq 4,$$

while $L(3) = 2 < 3$ and $L(2) = 1 < 2$, we conclude that for all $n \geq 4$ there exists at least one integer m such that m divides $\text{lcm}[m-1, \cdots, m-(n-1)]$.

Now, if we require that there exists precisely one integer $m \geq n$ dividing $\text{lcm}[m-1, \cdots, m-(n-1)]$ then we actually ask that there exists precisely one integer at least equal to n which divides $L(n)$, i.e., that integer would be $L(n)$. So, we're asking in this case for which $n \geq 4$ we have that the only divisor of $\text{lcm}[1, \ldots, n-1]$ at least equal to n is $L(n)$. We claim that in this case we must have that $n = 4$.

First of all, we have $L(4) = \text{lcm}[1, 2, 3] = 6$ and so indeed only 6 is at least equal to 4 and divides 6. Now, if $n \geq 5$, then both $(n-1)(n-2)$ and also $(n-2)(n-3)$ are greater than n and they divide $\text{lcm}[1, \ldots, n-1]$, which finishes our proof.

Problem 4. Find the minimum of

$$\max\{a+b+c, b+c+d, c+d+e, d+e+f, e+f+g\}$$

where the real numbers a, b, c, d, e, f, g vary among all the possible nonnegative solutions to the equation $a + b + c + d + e + f + g = 1$.

Solution. We have that

$$(a + b + c) + (d + e + f) + (e + f + g) \geq a + b + c + d + e + f + g = 1$$

and therefore, $M := \max\{a + b + c, b + c + d, c + d + e, d + e + f, e + f + g\} \geq \frac{1}{3}$. On the other hand, this minimum value of $\frac{1}{3}$ for M is attained in the case

$$a = \frac{1}{3},\ b = c = 0,\ d = \frac{1}{3},\ e = f = 0,\ g = \frac{1}{3}.$$