PUTNAM PRACTICE SET 9

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Problem 1. Find all real numbers a, b, c with the property that the equation

$$x^3 + ax^2 + bx + c = 0$$

has 3 real roots r_1, r_2, r_3 (not necessarily distinct) with the property that the equation

$$x^3 + a^3x^2 + b^3x + c^3 = 0$$

has the roots r_1^3, r_2^3, r_3^3.

Solution. So, we're asked to find a, b and c such that there exist distinct real numbers r_1, r_2 and r_3 such that

$$\begin{aligned} -a^3 &= (r_1 + r_2 + r_3)^3 = r_1^3 + r_2^3 + r_3^3 \\ b^3 &= (r_1r_2 + r_1r_3 + r_2r_3)^3 = r_1^3r_2^3 + r_1^3r_3^3 + r_2^3r_3^3 \\ -c &= r_1r_2r_3. \end{aligned}$$

The first equation yields

$$\begin{aligned} &0 \\ &= 3(r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_2r_3^2 + r_1r_3^2) + 6r_1r_2r_3 \\ &= 3(r_1r_2 + r_2r_3 + r_3r_1)(r_1 + r_2 + r_3) - 3r_1r_2r_3 \\ &= -3ab + 3c; \end{aligned}$$

so, $c = ab$. Now, the second condition yields similarly

$$\sum_{\substack{\sigma \text{ is a permutation} \\ \text{of } \{1,2,3\}}} x_{\sigma(1)}^3 x_{\sigma(2)}^2 x_{\sigma(3)} + 2r_1^2r_2^2r_3^2 = 0$$

and so,

$$\begin{aligned} &0 \\ &= (r_1^2r_2r_3 + r_1r_2^2r_3 + r_1r_2r_3^2)(r_1r_2 + r_2r_3 + r_3r_1) - r_1^2r_2^2r_3^2 \\ &= r_1r_2r_3(r_1 + r_2 + r_3) \cdot b - c^2 \\ &= (-c)(-a)b - c^2 \\ &= c(ab - c). \end{aligned}$$

Thus, the condition $c = ab$ is sufficient to guarantee that the solutions of $x^3 + a^3x^2 + b^3x + c^3 = 0$ are precisely r_1^3, r_2^3 and r_3^3, where r_1, r_2 and r_3 are the solutions of $x^3 + ax^2 + bx + c = 0$.

Problem 2. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-constant function with the property that for each $x, y > 0$ we have that $f(xy) = f(x)f(y)$. Find two functions $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying the properties:

- $h\left(\frac{x}{y}\right) = h(x)h(y) - g(x)g(y)$ for each $x, y > 0$; and
- $h(x) + g(x) = f(x)$ for each $x > 0$.

Solution. We note that $f(x \cdot 1) = f(x) \cdot f(1)$ and so we must have that $f(1) = 1$. Also, we obtain that $f(1/x) = 1/f(x)$. Now, using $y = x$ in the first relation above, we get

$$h(1) = h(x)^2 - g^2(x)$$

which we then combine with $h(x) + g(x) = f(x)$ in the special case $x = 1$ and therefore,

$$h(1) = h(1)^2 - g(1)^2 = (h(1) - g(1)) \cdot (h(1) + g(1)) = (h(1) - g(1)) \cdot f(1) = h(1) - g(1),$$

which yields that $g(1) = 0$ and in turn, $h(1) = 1$. Thus,

$$1 = h(1) = h(x)^2 - g(x)^2 = (h(x) - g(x)) \cdot (h(x) + g(x)) = (h(x) - g(x)) \cdot f(x)$$

and so, $h(x) - g(x) = 1/f(x)$, while $h(x) + g(x) = f(x)$. In conclusion,

$$h(x) = \frac{f(x)^2 + 1}{2f(x)} \text{ and } g(x) = \frac{f(x)^2 - 1}{2f(x)}.$$

Using the fact that $f(x/y) = f(x)/f(y)$, we verify that indeed,

$$h\left(\frac{x}{y}\right) = h(x)h(y) - g(x)g(y) \text{ for all } x \text{ and } y.$$

Problem 3. Let $x, y, z \in \mathbb{N}$ such that $xy - z^2 = 1$. Prove that there exist nonnegative integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$ and $z = ac + bd$.

Solution. First of all, we note the following: if $x, y, z \in \mathbb{N}$ satisfy $xy - z^2 = 1$, then

(i) $x \neq y$ and so, without loss of generality, we may assume $x < y$.
(ii) With the above convention regarding $x < y$, we then have either that $(x, y, z) = (1, 2, 1)$ or that $x < z < y$.

Indeed, if $x = y$ we would have $x^2 - z^2 = 1$ which has no solution in **positive** integers. Therefore, we must have $x \neq y$ and so, without loss of generality, we may assume $x < y$, which proves part (i) above.

Now, for the part (ii) above, we assume that $(x, y, z) \neq (1, 2, 1)$, which is equivalent with asking that $z > 1$. Now, since $y^2 > xy > z^2$, then we must have that $y > z$. If in addition, $x \geq z$, then $y > x \geq z$ and so,

$$xy - z^2 \geq yz - z^2 = z(y - z) \geq z > 1,$$

contradiction. So, indeed $y > z > x$, which proves part (ii).

We observe that if $z = 1$ (and so, $(x, y) = (1, 2)$ assuming $x < y$, as above), then the conclusion of our problem holds trivially with $a = 1$, $b = 0$, $c = d = 1$.

So, from now on, for any solution (x, y, z) of $xy - z^2 = 1$ we assume that $x < z < y$.

Now, in order to derive the desired conclusion, we argue by contradiction and therefore assume there exists a triple $(x_0, y_0, z_0) \in \mathbb{N}^3$ such that

- $x_0 y_0 - z_0^2 = 1$,
- there exist no nonnegative integers a_0, b_0, c_0, d_0 such that $x_0 = a_0^2 + b_0^2$, $y_0 = c_0^2 + d_0^2$ and $z_0 = a_0 c_0 + b_0 d_0$.

- z_0 is minimal among all such triples.

We claim then that $(x_0, x_0 + y_0 - 2z_0, z_0 - x_0)$ is another triplet satisfying the first two conditions above and clearly, contradicting the minimality of z_0 above. First of all, since $x_0, y_0, z_0 \in \mathbb{N}$, we have that $z_0 > x_0$ (as above in part (i)) and so, $x_0, z_0 - x_0 \in \mathbb{N}$; furthermore, $x_0 + y_0 - 2z_0 > 0$ because

$$(x_0 + y_0)^2 \geq 4x_0y_0 > 4z_0^2.$$

Now, regarding the first condition above:

$$\begin{aligned} & x_0(x_0 + y_0 - 2z_0) - (z_0 - x_0)^2 \\ &= x_0^2 + x_0y_0 - 2z_0x_0 - z_0^2 + 2z_0x_0 - x_0^2 \\ &= x_0y_0 - z_0^2 \\ &= 1, \end{aligned}$$

as claimed. As for the second condition above, if there were some nonnegative integers a_1, b_1, c_1, d_1 such that

$$\begin{aligned} x_0 &= a_1^2 + b_1^2 \\ x_0 + y_0 - 2z_0 &= c_1^2 + d_1^2 \\ z_0 - x_0 &= a_1c_1 + b_1d_1 \end{aligned}$$

then

$$z_0 = a_1c_1 + b_1d_1 + a_1^2 + b_1^2 = a_1(a_1 + c_1) + b_1(b_1 + d_1)$$

and

$$y_0 = c_1^2 + d_1^2 + a_1^2 + b_1^2 + 2a_1c_1 + 2b_1d_1 = (a_1 + c_1)^2 + (b_1 + d_1)^2,$$

thus delivering the desired contradiction.

Problem 4. Let $P, Q \in \mathbb{R}[x, y]$ be polynomials satisfying the following properties:

(A) for each $y_0 \in \mathbb{R}_{\geq 0}$, the functions $x \mapsto P(x, y_0)$ and $x \mapsto Q(x, y_0)$ are strictly increasing;
(B) for each $x_0 \in \mathbb{R}_{\geq 0}$, the function $y \mapsto P(x_0, y)$ is strictly increasing, while the function $y \mapsto Q(x_0, y)$ is strictly decreasing; and
(C) $P(x, 0) = Q(x, 0)$ for each $x \in \mathbb{R}_{\geq 0}$ and also, $P(0, 0) = 0$.

Prove the following:

(1) for each real numbers $0 \leq b \leq a$, there exists a unique pair (x_0, y_0) of nonnegative real numbers with the property that $P(x_0, y_0) = a$ and $Q(x_0, y_0) = b$.
(2) if $0 \leq a < b$, then there exist no nonnegative real numbers x_0 and y_0 such that $P(x_0, y_0) = a$ and $Q(x_0, y_0) = b$.

Solution. For each $a \geq 0$, we let $x_a \in [0, +\infty)$ be the unique real number with the property that $P(x_a, 0) = a$. Note that the existence uniqueness of x_a follows from the following facts:

- the function $x \mapsto P(x, 0)$ is strictly increasing and since $P(x, y)$ is a polynomial, then the function $x \mapsto P(x, 0)$ is a polynomial as well, and therefore, $\lim_{x \to +\infty} P(x, 0) = +\infty$.
- $P(0, 0) = 0$ and so, the continuous function $x \mapsto P(x, 0)$ must take all posible nonnegative real values as we let x vary in $[0, +\infty)$.

From now on, we let a be a fixed nonnegative real number. For each $x \in [0, x_a]$ we claim that there exists a unique real number, denoted by $f_a(x)$ (which is thus a well-defined function) such that $P(x, f_a(x)) = a$. Indeed, for each given $\tilde{x} \in [0, x_a]$, we know that

- the function $y \mapsto P(\tilde{x}, y)$ is strictly increasing and since $P(x, y)$ is a polynomial, then the function $y \mapsto P(\tilde{x}, y)$ is a polynomial as well, and therefore, $\lim_{y\to+\infty} P(\tilde{x}, y) = +\infty$.
- $P(\tilde{x}, 0) \le P(x_a, 0) = a$ since the function $x \mapsto P(x, 0)$ is a strictly increasing function.

In particular, using also the choice of x_a, we have that $f_a(x_a) = 0$.

Claim. The function f_a is continuous on $[0, x_a]$.

Proof of Claim. This follows by proving the following two properties:

(i) f_a satisfies the Intermediate Value Theorem (IVT) on any subinterval of $[0, x_a]$.

(ii) f_a is strictly decreasing.

We see that $f_a(\tilde{x}_2) < f_a(\tilde{x}_1)$ whenever $0 \le \tilde{x}_1 < \tilde{x}_2 \le x_a$, because we have that

$$\begin{aligned} &P\left(\tilde{x}_2, f_a(\tilde{x}_2)\right) \\ &= a \\ &= P\left(\tilde{x}_1, f_a(\tilde{x}_1)\right) \ \text{(by the definition of } f_a\text{)} \\ &< P\left(\tilde{x}_2, f_a(\tilde{x}_1)\right) \ \text{(since } x \mapsto P\left(x, f_a(\tilde{x}_1)\right) \text{ is strictly increasing)} \end{aligned}$$

and finally (once again) $y \mapsto P\left(\tilde{x}_2, y\right)$ is strictly increasing.

Now, in order to see that f_a satisfies the IVT on each subinterval of $[0, x_a]$, we observe that if

$$f_a(\tilde{x}_1) = b_1 > b_2 = f_a(\tilde{x}_2)$$

for some $0 \le \tilde{x}_1 < \tilde{x}_2 \le x_a$ and moreover, $c \in (b_2, b_1)$, then we have that

$$\begin{aligned} &P\left(\tilde{x_2}, c\right) \\ &> P\left(\tilde{x_2}, b_2\right) \ \text{(since } y \mapsto P\left(\tilde{x}_2, y\right) \text{ is strictly increasing)} \\ &= P\left(\tilde{x}_2, f_a(\tilde{x}_2)\right) \\ &= a \\ &= P\left(\tilde{x}_1, f_a(\tilde{x}_1)\right) \\ &= P\left(\tilde{x}_1, b_1\right) \\ &> P\left(\tilde{x}_1, c\right) \ \text{(since } y \mapsto P(\tilde{x}_1, y) \text{ is strictly increasing.)} \end{aligned}$$

Then, using the fact that $x \mapsto P(x, c)$ is a continuous function and

$$P\left(\tilde{x_1}, c\right) < a < P\left(\tilde{x_2}, c\right),$$

we conclude (once again by the IVT) the existence of some $\tilde{x} \in (\tilde{x}_1, \tilde{x}_2)$ such that $f_a(\tilde{x}) = c$.

Finally, in order to conclude the proof of our claim, we are using the fact that a strictly monotone function g on any interval $[c, d]$ which satisfies the IVT must actually be continuous. The point is that for any $t \in (c, d)$ (an almost identical argument applies also to the two endpoints of the interval), we have well-defined lateral limits:

$$L_t^- := \lim_{x\to t^-} g(x) \text{ and } L_t^+ := \lim_{x\to t^+} g(x)$$

because $g(x)$ is a strictly monotone function. Now, since g satisfies the IVT (on any subinterval of $[c,d]$), then we see that

$$L_t^- = g(t) = L_t^+$$

since otherwise the IVT would fail on a small interval centered around t.

In conclusion, our function f_a is indeed a continuous function, which concludes the proof of the given **Claim**.

Now, going back to our original setting, we let the function $h : [0, x_a] \longrightarrow \mathbb{R}$ given by $h(x) := Q(x, f_a(x))$. Because Q is a polynomial, while f_a is continuous, we conclude that also the function h is continuous. We have

$$\begin{aligned} & h(x_a) \\ &= Q(x_a, f_a(x_a)) \\ &= Q(x_a, 0) \text{ (since } f_a(x_a) = 0 \text{ by our choice of } x_a \text{ and of } f_a) \\ &= P(x_a, 0) \text{ (by our hypothesis)} \\ &= a \\ &\geq 0 \\ &= Q(0,0) \\ &\geq Q(0, f_a(0)) \text{ (since } f_a(0) \geq 0 \text{ and } y \mapsto Q(0,y) \text{ is decreasing)} \\ &= h_a(0) \text{ (by definition of the function } h_a). \end{aligned}$$

So, using again the IVT, for any $0 \leq b \leq a$, there exists some $\tilde{x} \in [0, x_a]$ such that $h(\tilde{x}) = b$, i.e., letting $\tilde{y} := f_a(\tilde{x})$ we have that

$$Q(\tilde{x}, \tilde{y}) = b \text{ and } P(\tilde{x}, \tilde{y}) = a.$$

Now, we claim that $\tilde{x}$ is unique with the property that $h(\tilde{x}) = b$. Indeed, this claim follows from the fact that Q is strictly increasing in the first variable and strictly decreasing in the second variable, while f_a is strictly decreasing, and therefore, for any $0 \leq \tilde{x}_1 < \tilde{x}_2 \leq x_a$, we have that

$$h(\tilde{x}_1) = Q(\tilde{x}_1, f_a(\tilde{x}_1)) < Q(\tilde{x}_2, f_a(\tilde{x}_1)) < Q(\tilde{x}_2, f_a(\tilde{x}_2)) = h(\tilde{x}_2).$$

Finally, we cannot have any simultaneous solution (x_0, y_0) in nonnegative real numbers for the equations

$$P(x_0, y_0) = a \text{ and } Q(x_0, y_0) = b$$

if $a < b$ since then we would have

$$a = P(x_0, y_0) > P(x_0, 0) = Q(x_0, 0) > Q(x_0, y_0) = b,$$

contradiction.