PUTNAM PRACTICE SET 7

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Problem 1. Let $a, b, c, d \in \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = 1 - a\cos(x) - b\sin(x) - c\cos(2x) - d\sin(2x)$. If $f(x) \geq 0$ for each $x \in \mathbb{R}$, then prove that $a^2 + b^2 \leq 2$ and also that $c^2 + d^2 \leq 1$.

Solution. Since $\sin(x+\pi) = -\sin(x)$ and $\cos(x+\pi) = -\cos(x)$, we observe that

$$g(x) := f(x) + f(x+\pi) = 2 - 2c\cos(2x) - 2d\sin(2x);$$

so, for each $x \in \mathbb{R}$ we must have that $c\cos(2x) + d\sin(2x) \leq 1$.

We have that (the Cauchy-Schwarz Inequality)

$$c\cos(2x) + d\sin(2x) \leq \sqrt{(c^2+d^2)\cdot(\cos^2(2x)+\sin^2(2x))} = \sqrt{c^2+d^2}.$$

This inequality comes from the classical inequality:

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

which reduces to

$$0 \leq (x_1y_2 - x_2y_1)^2.$$

On the other hand, we can find $x_0 \in \mathbb{R}$ such that (if $c, d \neq 0$)

$$\cos(2x_0) = \frac{c}{\sqrt{c^2+d^2}} \text{ and } \sin(2x_0) = \frac{d}{\sqrt{c^2+d^2}}$$

since $\left(\frac{c}{\sqrt{c^2+d^2}}\right)^2 + \left(\frac{d}{\sqrt{c^2+d^2}}\right)^2 = 1$. So,

$$g(x_0) = 2\cdot\left(1 - \sqrt{c^2+d^2}\right)$$

and because $g(x_0) \geq 0$, we must have that $c^2 + d^2 \leq 1$.

Similarly, noting that $\cos(x+\pi/2) = -\sin(x)$ and $\sin(x+\pi/2) = \cos(x)$, we see that

$$h(x) := f(x) + f\left(x + \frac{\pi}{2}\right) = 2 - a\left(\cos(x) - \sin(x)\right) - b\left(\sin(x) + \cos(x)\right).$$

Furthermore,

$$\cos(x) - \sin(x) = \sin(x+\pi/2) - \sin(x) = 2\sin(\pi/4)\cos(x+\pi/4) = \sqrt{2}\cdot\cos\left(x + \frac{\pi}{4}\right)$$

and

$$\sin(x) + \cos(x) = \sin(x) + \sin(x+\pi/2) = 2\sin(x+\pi/4)\cdot\cos(\pi/4) = \sqrt{2}\cdot\sin\left(x + \frac{\pi}{4}\right).$$

So, $f(x) = 2 - a\sqrt{2}\cos(x+\pi/4) - b\sqrt{2}\sin(x+\pi/4)$ and we can find $x_1 \in \mathbb{R}$ such that (assuming not both a and b are equal to 0)

$$\cos\left(x_1 + \frac{\pi}{4}\right) = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\left(x_1 + \frac{\pi}{4}\right) = \frac{b}{\sqrt{a^2+b^2}}.$$

Then $h(x_1) = 2 - \sqrt{2}\cdot\sqrt{a^2+b^2} \geq 0$ yields the inequality $a^2 + b^2 \leq 2$.

It is important to note that we do not claim that the conditions $a^2+b^2 \le 2$ and $c^2+d^2 \le 1$ are also sufficient to imply that $f(x) \ge 0$ for all $x \in \mathbb{R}$.

Problem 2. Find all positive integers n for which there exist nonzero polynomials $f, g \in \mathbb{Z}[x_1, \dots, x_n]$ such that

$$(x_1 + x_2 + \cdots + x_n) \cdot f(x_1, \dots, x_n) = g(x_1^2, \dots, x_n^2).$$

Solution. We'll show that for each n there are such polynomials f and g. Indeed, let

$$f(x_1, \dots, x_n) = \prod_{\substack{\epsilon_1, \dots, \epsilon_n \in \{-1,1\} \\ (\epsilon_1, \dots, \epsilon_n) \neq (1,1,\cdots,1)}} (\epsilon_1 x_1 + \cdots + \epsilon_n x_n)\,;$$

this is a polynomial of degree $2^n - 1$ and we claim that

$$h(x_1, \dots, x_n) := (x_1 + \cdots + x_n) \cdot f(x_1, \dots, x_n)$$

is a polynomial of the form $g(x_1^2, \dots, x_n^2)$. Indeed, for each $i = 1, \dots, n$, if we let σ_i be the automorphism of $\mathbb{Z}[x_1, \dots, x_n]$ given by

$$\sigma_i(x_i) = -x_i \text{ and } \sigma_i(x_j) = x_j \text{ for each } j \neq i,$$

then we see that $\sigma_i(h) = h$. Therefore, each monomial of h contains x_i at an even power. So, repeating this argument for each $i = 1, \dots, n$, we conclude that indeed,

$$h(x_1, \dots, x_n) = g(x_1^2, \dots, x_n^2) \text{ for some } g \in \mathbb{Z}[x_1, \dots, x_n].$$

Problem 3. Let $n \ge 2$ be a positive integer and let S_n be the set of all integers of the form $1 + kn$ for some $k \in \mathbb{N}$. We say that a number $m \in S_n$ is indecomposable if there exist no $x, y \in S_n$ such that $m = xy$. Prove that there exists some $s \in S_n$ which can be written in at least two distinct ways as a product of indecomposable numbers from S_n (note that two decompositions consisting of precisely the same indecomposable numbers, but appearing in a different order are considered to be the same decomposition).

Solution. Let p be a prime number larger than n which is not in in S_n. The existence of such a prime number p follows from the classical argument assuming first that there exist finitely many prime numbers $p_1, \dots, p_\ell$ which are not in S_n, then we consider $N := n \cdot \prod_{i=1}^{\ell} p_i - 1$; then $N > 1$ and it is not divisible by any of the numbers p_i but also, it cannot be divisible only by prime numbers contained in S_n. So, indeed, there exist infinitely many prime numbers not contained in S_n and we simply pick a prime number $p > n$ not contained in S_n.

We let k be the smallest positive integer such that $p^k \equiv 1 \pmod{n}$ and therefore, k is the smallest positive integer such that $p^k \in S_n$. Clearly, p^k is indecomposable in S_n and also, clearly, $k \ge 2$.

Now, let $m \in \mathbb{N}$ be the smallest positive integer such that $pm \equiv 1 \pmod{n}$; clearly, $m \in \{2, \dots, n-1\}$. Also, clearly, pm is indecomposable since otherwise there would exist some $1 \le i < m$ such that $i \cdot p \equiv 1 \pmod{n}$, a contradiction.

We claim that $M := p^k m^k$ can be written in at least two ways as a product of indecomposable numbers from S_n. Indeed, M is the product of k copies of pm and pm is indecomposable. On the other hand, we can write M as a product of p^k (which is indecomposable) and the product of indecomposables which make up the number m^k (which is in S_n because $p^k \cdot m^k \equiv 1 \pmod n$ and $p^k \equiv 1 \pmod n$). Finally, we note that pm is not p^k because

$$p^k \geq p^2 > p \cdot n > pm.$$

Problem 4. Let n be an integer ≥ 2. We define two sequences $\{x_i\}_{1\leq i\leq n}$ and $\{y_i\}_{1\leq i\leq n}$ given by:

$$x_1 = n,\ y_1 = 1,\ x_{i+1} = \left[\frac{x_i + y_i}{2}\right] \text{ and } y_{i+1} = \left[\frac{n}{x_{i+1}}\right],$$

where $[z]$ is the integer part of z for each real number z. Prove that

$$\min_{i=1}^{n} x_i = [\sqrt{n}].$$

Solution. We have that

$$\frac{x_i + y_i}{2} \leq \frac{x_i + \frac{n}{x_i}}{2} < \frac{x_i + y_i + 1}{2}$$

because $y_i = \left[\frac{n}{x_i}\right] \leq \frac{n}{x_i} < y_i + 1$. Now, regardless whether $x_i + y_i$ is even or odd, we get that

$$x_{i+1} = \left[\frac{x_i + y_i}{2}\right] = \left[\frac{x_i + \frac{n}{x_i}}{2}\right].$$

So, because $x_i + \frac{n}{x_i} \geq 2\sqrt{n}$ for each i (and also $x_1 = n$), then we conclude that $x_i \geq [\sqrt{n}]$ for all $i \geq 1$; hence

$$\min_{i=1}^{n} x_i \geq \left[\sqrt{n}\right].$$

Claim. If $x_i \geq [\sqrt{n}] + 1$, then $x_i > x_{i+1}$.

*P*roof of Claim. Since $x_i \geq [\sqrt{n}] + 1 > \sqrt{n}$, then $\frac{n}{x_i} - x_i < 0$ and so, $\left[\frac{\frac{n}{x_i} - x_i}{2}\right] \leq -1$; then we have that

$$x_{i+1} - x_i = \left[\frac{\frac{n}{x_i} - x_i}{2}\right] \leq -1,$$

as desired.

So, the above claim proves that starting from $x_1 = n$, then in less than $n - [\sqrt{n}] + 1$ steps we reach $x_i = [\sqrt{n}]$.