PUTNAM PRACTICE SET 6

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Problem 1. Find the largest possible integer which is the product of finitely many positive integers whose sum equals 2018.

Solution. Let $x_1, \dots, x_r$ be positive integers whose sum is 2018 and which have the largest possible product.

First we notice that if $x_r \ge 4$, then replacing x_r by $x'_r = 2$ and $x'_{r+1} = x_r - 2$, while $x'_i = x_i$ for each $i \le r-1$ leads to a larger product. So, this means that each x_i is less than 4.

Secondly, we cannot have $x_r = 1$ since then replacing x_{r-1} by $x'_{r-1} = x_{r-1} + 1$ and keeping $x'_i = x_i$ for each $i \le r-2$ would lead to a sequence $x'_1, \dots, x'_{r-1}$ whose sum is 2018 but whose product is larger than for the product of the original numbers x_i.

So, we conclude that each $x_i \in \{2, 3\}$. Now, if we were to have three of the x_i's equal to 2, we could replace them with two numbers equal to 3 and the product would only increase. Therefore, we have only one or two nmbers equal to 2 and all the rest of the numbers equal 3. Since $2018 \equiv 2 \pmod 3$, this means $x_1 = 2$ and $x_i = 3$ for each $i = 2, \dots, r$; clearly, since $2 + 3r = 2018$, then we must have $r = 672$. So, the largest product of the numbers adding up to 2018 is $2 \cdot 3^{672}$.

Problem 2. Let $P \in \mathbb{R}[x]$ be a polynomial with the property that $P(x) > 0$ for each positive real number x. Then prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ with all coefficients nonnegative, such that $P = \frac{Q_1}{Q_2}$.

Solution. First of all, since $P(x) > 0$ for all $x > 0$, we conclude that its leading coefficient must be positive; so, without loss of generality we may assume from now on that $P(x)$ is monic since its leading coefficient can be absorbed in $Q_1(x)$.

Second, we know that $P(x)$ is a product of linear polynomials of the form $x + r_i$ for some nonnegative real numbers r_i and perhaps also a product of unfactorable quadratics (over $\mathbb{R}$), i.e., quadratics of the form $x^2 + a_i x + b_i$ where $a_i^2 < 4b_i$. So, it suffices to prove that each polynomial of the form

$$x + r_i \text{ for some } r_i \ge 0, \text{ and}$$

$$x^2 + a_i x + b_i \text{ where } a_i^2 < 4b_i.$$

is of the form $\frac{Q_{1,i}(x)}{Q_{2,i}(x)}$ where each $Q_{1,i}, Q_{2,i}$ are polynomials with nonnegative real coefficients. Clearly, this statement holds for polynomials of the form $x + r_i$; so, we're left to analyze the case of quadratic polynomials. In this latter case, we let

$$f(x) := x^2 + ax + b$$

such a quadratic polynomial with $a^2 < 4b$; then we let $b := r^2$ for some positive real number r and then we let $t \in [0, \pi]$ such that $a = -2r\cos(t)$. Our goal is to find some polynomials $g_1(x)$ and $g_2(x)$ with nonnegative real coefficients such that $f(x) = \frac{g_1(x)}{g_2(x)}$.

If $t \in [\pi/2, \pi]$, then we are done (simply take $g_1(x) := f(x)$ and $g_2(x) := 1$). Now, if $t \in [0, , \pi/2)$ (i.e., $\cos(t) > 0$ and implicitly, $a < 0$), we observe that

$$f(x) \cdot (x^2 - ax + b) = x^4 - (a^2 - 2b)x^2 + b^2 = x^4 - 2r^2 \cos(2t) + r^4.$$

Then we repeat our analysis and so, if $2t \in [\pi/2, \pi]$, then we are done since then $\cos(2t) \le 0$. Now, if $2t \in [0, \pi/2)$ (and so, implicitly, $a^2 > 2b$), then we repeat the construction and get:

$$f(x) \cdot (x^2 - ax + b) \cdot (x^4 + (a^2 - 2b)x^2 + b^2) = x^8 - 2r^4 \cos(4t)x^4 + r^8.$$

Eventually, there must exist a first positive nonenegative integer i_0 such that $2^{i_0}t \in [\pi/2, \pi]$ and for that i_0, we have that the corresponding polynomial

$$x^{2^{i_0+1}} - 2r^{2^{i_0}} \cos\left(2^{i_0}t\right) x^{2^{i_0}} + r^{2^{i_0+1}}$$

has all its coefficients nonenegative and we reached this polynomial by multiplying $f(x)$ by polynomials which were themselves with nonnegative coefficients.

Problem 3. Prove that there exist infinitely many $n \in \mathbb{N}$ with the property that 7^n contains in its decimal expansion 2018 consecutive digits equal to 0.

Solution. The point is that $\gcd(7, 10)$ and so, Euler's Theorem guarantees that

$$7^{5^{2018} \cdot 2^{2020}} \equiv 7^{\phi\left(10^{2019}\right)} \equiv 1 \pmod{10^{2019}},$$

thus showing that $7^{5^{2018} \cdot 2^{2020}}$ ends with the digit 1 and it has 2018 digits of 0 preceding that last digit.

Problem 4. Let $a \in (0, 1)$ be a real number. We consider the function $f : (0, 1] \longrightarrow (0, 1]$ given by:

$$f(x) = x + 1 - a \text{ if } 0 < x \le a \text{ and } f(x) = x - a \text{ if } a < x \le 1.$$

Prove that for any interval $I \subseteq (0, 1]$, there exists a positive integer n such that $f^{\circ n}(I) \cap I \neq \emptyset$.

Solution 1. We note that f is a bijection map sending $(0, 1]$ into itself. Also, we claim that for any interval $J \subseteq (0, 1]$, we have that $f(J)$ is also a union of intervals whose sums of their lengths equals the length of J. This is proven easily by considering the three cases:

Case 1. $J \subseteq (0, a]$. In this case, $f(J)$ is an interval of the same length as J contained in $(1 - a, 1]$.

Case 2. $J \subseteq (a, 1]$. In this case, $f(J)$ is an interval of the same length as J contained in $(0, 1 - a]$.

Case 3. $J = (\alpha, \beta]$ (or any other choice of including or not any of the two endpoints) for some $0 \le \alpha < a < \beta \le 1$. Iin this case, $f(J) = (\alpha+1-a, 1] \cup (0, \beta-a]$ whose length is

$$1 - (\alpha + 1 - a) + (\beta - a) - 0 = \beta - \alpha, \text{ as claimed.}$$

Now, if $f^n(I) \cap I = \emptyset$, then we claim that $f^i(I) \cap f^j(I) = \emptyset$ for any integers $i > j \ge 0$. Indeed, using the fact that f is a bijection on $(0, 1]$ (and therefore, f^m is a bijection for each $m \in \mathbb{N}$), we get that if there exists some $x \in f^i(I) \cap f^j(I)$, then letting $y \in (0,1]$ be the unique real number such that $f^j(y) = x$, then we would have that $y \in I \cap f^{i-j}(I)$, contradiction. (Note that we do not claim that y is fixed by f^{i-j}, however we know that $x = f^j(y) \in f^j(f^{i-j}(I))$ and f^j is a bijection, thus showing that $y \in f^{i-j}(I)$.) But then we would have an infinite sequence of unions of intervals $f^n(I)$, each one of them of total length equal to the length of I and all these intervals would fit into the interval $(0,1]$, which is a contradiction. So, indeed there must be some $n \in \mathbb{N}$ such that $f^n(I) \cap I \neq \emptyset$.

*S*olution 2. We notice that from our definition of the function f, we have that for any real number x, we have that $f(x) - x + a \in \mathbb{Z}$. By induction, we prove that $f^n(x) - x + na \in \mathbb{Z}$ for each $x \in (0,1]$. Indeed, assuming that there exists some $p_n(x) \in \mathbb{Z}$ (i.e., an integer depending on x) such that

$$f^n(x) = x - na + p_n$$

then we compute

$$f^{n+1}(x) = f(x-na+p_n(x)) = x-na+p_n(x)-a+p_1(x-na+p_n(x)) \in (x-(n+1)a)+\mathbb{Z},$$

where $p_1(x) := f(x) - (x - a)$ (and more generally, $p_n(x) := f^n(x) - (x - na)$). So, indeed, $f^n(x) - (x - na) \in \mathbb{Z}$ for each $n \in \mathbb{N}$ and for each $x \in (0,1]$.

Now, for any given interval I we claim that there must exist some $x \in I$ such that also $x \in f^n(I)$, i.e., there exists some $y \in I$ such that

$$x = f^n(y) = y - na + p_n(y).$$

So, $na - p_n(y) = y - x$, i.e., for any $\epsilon > 0$, there exists some positive integer n and some integer q_n such that $na - q_n \in (-\epsilon, \epsilon)$. The conclusion follows from a classical argument looking at the fractional part of na as we vary $n \in \mathbb{N}$ and note that for some N sufficiently large (anything larger than $1/\epsilon$ would work) we must have two distinct integers $N \geq i > j \geq 0$ such that $|\{ia\} - \{ja\}| < \epsilon$ and so, $(i-j)a - q \in (-\epsilon, \epsilon)$, where $q = [ia] - [ja]$ (the difference of their corresponding integer parts).

Problem 5. Find (with proof) all possible function $f : \mathbb{N} \longrightarrow \mathbb{N}$ with the property that $f(n+1) > f(f(n))$ for each $n \in \mathbb{N}$.

Solution. We will prove that there is only one such function, which is $f(n) = n$ for each $n \in \mathbb{N}$.

First we prove by induction on k that for each $n \geq k$, we have that $f(n) \geq k$. The base case $k = 1$ is obvious. So, assuming that we prove $f(n) \geq k$ for each $n \geq k$, next we derive that $f(n) \geq k+1$ for each $n \geq k+1$. Indeed, for any $n \geq k$, we have that

$$\begin{aligned} & f(n+1) \\ & > f(f(n)) \text{ by the main hypothesis} \\ & = f(m) \text{ for some } m \geq k \text{ since } n \geq k \text{ and using the inductive hypothesis} \\ & \geq k \text{ again by the inductive hypothesis.} \end{aligned}$$

So, indeed, $f(n+1) \geq k+1$ for each $n \geq k$, which concludes the proof for our claim that $f(n) \geq k$ whenever $n \geq k$ for any given $k \in \mathbb{N}$.

Now, assume there exists some $n \in \mathbb{N}$ such that $f(n) > n$ and we will derive a contradiction, which will conclude our proof that the only function is the one satisfying $f(n) = n$ for each $n \in \mathbb{N}$. So, let n_1 be the smallest positive integer n such that $f(n) > n$. Clearly, we cannot have $f(n_1) = n_1 + 1$ since then

$$f(n_1 + 1) > f(f(n_1)) = f(n_1 + 1), \text{ contradiction.}$$

Also, since n_1 is the smallest such positive integer, then it must be that for each positive integer $n < n_1$, we have that $f(n) = n$. Now, for each $n > n_1$, we have that $f(n) > f(f(n-1))$ and moreover, $f(n-1) \geq n_1$ since $n > n_1$ and $f(k) \geq k$ for each $k \in \mathbb{N}$. Now, if $f(n-1) = n_1$, we note that it cannot be that $n - 1 \geq n_1 + 1$ since then we would have that $f(n-1) \geq n_1+1$, a contradiction. So, if $f(n-1) = n_1$ then we would get that $n - 1 = n_1$, which is again a contradiction since our assumption yields that $f(n_1) > n_1$. In conclusion, we must have that $f(n-1) > n_1$. So, this means that our hypothesis that $f(n_1) > n_1$ yields the following property: for each $n > n_1$, there exists some $m > n_1$ such that $f(n) > f(m)$ (more precisely, $m = f(n-1)$). But this would mean that the set of positive integers

$$\{f(n_1 + 1), f(n_1 + 2), \cdots, \cdots\}$$

does not have a minimal element, which is impossible. So, indeed, we must have that $f(n) = n$ for each $n \in \mathbb{N}$.