PUTNAM PRACTICE SET 5

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Problem 1. Let $f(x) = x^2 - 2$. For each $n \in \mathbb{N}$, we let $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n times). Prove that for each $n \in \mathbb{N}$ there exist 2^n real numbers x such that $f^{\circ n}(x) = x$.

Solution. This follows using a table of signs first for $P^{\circ n}(x)$ (which is a polynomial of degree 2^n); note that we denote by $P^{\circ n}$ the n-th iterate of the polynomial $P(X)$. The point is that $P(0) = -2$ and also $P(-2) = P(2) = 2$. This means that the equation $P^{\circ n}(x) = 0$ has precisely 2^n roots $x_{n,i}$ for $i = 1, \dots, 2^n$ which are all between -2 and 2; this is proven by induction on n and furthermore, these roots alternate with the roots of $P^{\circ(n-1)}(x) = 0$. These facts:

- $-2 < x_{n,1} < x_{n,2} < \cdots < x_{n,2^{n-1}} < 0 < x_{n,2^{n-1}+1} < \cdots < x_{n,2^n} < 2$;
- $P(0) = -2$, while $P(-2) = P(2) = 2$

yield the conclusion.

Problem 2. Prove that there exists an infinite set S of points on the unit circle of radius 1 with the property that the distance between any two points from the set S is a rational number.

Solution. We need to find points x_i on the unit circle at angles α_i (starting with $\alpha_0 = 0$) such that $\sin((\alpha_i - \alpha_j)/2)$ is a rational number for each i and j. So, it suffices to have $\sin(\alpha_i/2)$ and $\cos(\alpha_i/2)$ both rational numbers. This can be achieved since there exist infinitely many points with rational coordinates on the unit circle:

$$\left(\frac{2u}{u^2+1}, \frac{1-u^2}{u^2+1}\right) \text{ for any } u \in \mathbb{Q}.$$

Problem 3. Consider the sequence $\{x_n\}_{n\geq 0}$ given by:

$$x_0 = 5 \text{ and } x_{n+1} = x_n + \frac{1}{x_n} \text{ for all } n \geq 0.$$

Prove that $45 < x_{1000} < 45.1$.

Solution. We have

$$x_{n+1}^2 = x_n^2 + 2 + \frac{1}{x_n^2}$$

and so, $x_{n+1}^2 > x_n^2 + 2$ for each $n \geq 0$, which yields that

$$x_{1000}^2 > x_0^2 + 2 \cdot 1000 = 2025 = 45^2;$$

this provides the first inequality. Now, for the second inequality, we use that

$$x_{1000}^2 = 2025 + \sum_{k=0}^{1000} \frac{1}{x_k^2}$$

and moreover $x_k^2 > 25 + 2k$, which means that

$$\begin{aligned} x_{1000} & \\ &= 2025 + \sum_{k=0}^{1000} \frac{1}{x_k^2} \\ &< 2025 + \sum_{k=0}^{1000} \frac{1}{25+2k} \\ &< 2025 + \frac{1}{2025} + \sum_{k=0}^{39} \frac{25}{25+50k} \\ &< 2025.0005 + \sum_{k=0}^{39} \frac{1}{1+2k} \\ &< 2026.0005 + \sum_{k=0}^{12} \frac{3}{3+6k} \\ &< 2026.002 + \sum_{k=0}^{12} \frac{1}{1+2k} \\ &< 2027.01 + \sum_{k=0}^{3} \frac{3}{3+6k} \\ &< 2028.01 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} \\ &< 2029.01 \\ &< 2025 + 0.01 + 5 \\ &< 45.1^2 \end{aligned}$$

Problem 4. Let $n \in \mathbb{N}$ and let $a_0, a_1, \dots, a_{n+1} \in \mathbb{R}$ such that $a_0 = a_{n+1} = 0$ and $|a_{k-1} - 2a_k + a_{k+1}| \le 1$ for each $k = 1, \dots, n$. Prove that for each $k = 0, \dots, n+1$, we have $|a_k| \le \frac{k(n+1-k)}{2}$.

Solution. Writing

$$\begin{aligned} a_0 - 2a_1 + a_2 &= b_1 \\ a_1 - 2a_2 + a_3 &= b_2 \\ &\cdots\cdots\cdots \\ a_{n-1} - 2a_n + a_{n+1} &= b_n \end{aligned}$$

coupled with the information that $a_0 = a_{n+1} = 0$ yields a system of n equations with n unknowns $a_1, \dots, a_n$. So, we can solve it. We find a_k by multiplying the second equation by 2 and add it to the first equation and then add to it the third equation multiplied by 3, and so on, until multiplying the k-th equation by k and add it to our sum; we get

$$-(k+1)a_k + ka_{k+1} = b_1 + 2b_2 + \cdots + kb_k.$$

Then we start eliminating the variables starting with the last equation and proceed similarly, i.e., add the last equation with twice the next to the last equation and also add to the sum three times the equation involving b_{n-2}, four times the equation involving b_{n-3} and so on, until we add $(n-k)$ times the equation involving b_{k+1} and get

$$-(n-k+1)a_{k+1} + (n-k)a_k = b_n + 2b_{n-1} + \cdots + (n-k)b_{k+1}.$$

We solve for a_k from the above two equations (multiplying the first by $(n-k+1)$ and then add it to k-times the second equation) and we get

$$-(n+1)a_k = (n-k+1)\cdot\sum_{j=1}^{k} jb_j + k\cdot\sum_{j=1}^{n-k} jb_{n+1-j}.$$

Because $|b_j| \le 1$ for each $j = 1, \dots, n$, we conclude that indeed $|a_k| \le \frac{(n-k+1)k}{2}$, as desired.

Problem 5. A sequence $\{x_n\}_{n\ge 0}$ is defined as follows:

$$x_0 = 2,\ x_1 = \frac{5}{2} \text{ and for each } n \ge 1, \text{ we have } x_{n+1} = x_n(x_{n-1}^2 - 2) - x_1.$$

Prove that for each $n \in \mathbb{N}$, we have that the integer part of x_n (denoted by $[x_n]$) equals $2^{\frac{2^n-(-1)^n}{3}}$.

Solution. We try to find the general form of our sequence x_n by writing it as $x_n = v_n + v_n^{-1}$ for a sequence v_n (which we will solve next). The motivation for looking for such a form for our sequence $\{x_n\}_{n\ge 0}$ comes from solving for few terms of it:

$$x_0 = 2,\ x_1 = \frac{5}{2},\ x_2 = \frac{5}{2} = 2 + \frac{1}{2},\ x_3 = \frac{65}{8} = 8 + \frac{1}{8},\ x_4 = 32 + \frac{1}{32}\cdots$$

So, the general recurrence formula yields

$$v_{n+1} + \frac{1}{v_{n+1}} = v_n v_{n-1}^2 + \frac{1}{v_n v_{n-1}^2} + \frac{v_n}{v_{n-1}^2} + \frac{v_{n-1}^2}{v_n} - \frac{5}{2}.$$

Now, if we were to have $v_n = v_n \cdot v_{n-1}^2$ for each $n \ge 1$ and also, if we were to have $\frac{v_n}{v_{n-1}^2} \in \left\{2, \frac{1}{2}\right\}$, then we would achieve both

$$\frac{v_n}{v_{n-1}^2} + \frac{v_{n-1}^2}{v_n} = \frac{5}{2}$$

and also,

$$v_{n+1} + \frac{1}{v_{n+1}} = v_n v_{n-1}^2 + \frac{1}{v_n v_{n-1}^2}.$$

So, again looking at the first few terms computed above in the sequence x_n, we get the idea that $v_n = 2^{y_n}$. In this case, the equality $v_{n+1} = v_n \cdot v_{n-1}^2$ yields

$$y_{n+1} = y_n + 2y_{n-1},$$

while the relation $\frac{v_n}{v_{n-1}^2} \in \left\{2, \frac{1}{2}\right\}$ yields

$$y_n - 2y_{n-1} \in \{-1, 1\}.$$

The linear recurrence formula satisfied by $\{y_n\}$ yields that $y_n = \alpha\cdot 2^n + \beta\cdot(-1)^n$ for some constants α and β, while the relation $y_n - 2y_{n-1} \in \{-1, 1\}$ yields that $\beta = \pm\frac{1}{3}$. Using the information for the first values of y_n (based on our computation above) as $y_1 = y_2 = 1$, $y_3 = 3$, $y_4 = 5$, yields that $y_n = \frac{2^n-(-1)^n}{3}$ for each $n \ge 0$; note that $2^n \equiv (-1)^n \pmod 3$ and so, the exponent y_n is indeed always a nonnegative integer. In conclusion, we get that

$$x_n = 2^{\frac{2^n-(-1)^n}{3}} + 2^{\frac{(-1)^n-2^n}{3}}$$

and since the second term above is always in the interval $[0,1)$, while the first term is always an integer, we derive the desired conclusion.

Problem 6. Let $m \in \mathbb{N}$. We consider the m-by-$2m$ matrix

$$A = (a_{i,j})_{\substack{1\le i\le m\\ 1\le j\le 2m}}$$

with the property that each entry $a_{i,j}$ is either -1, 0, or 1. Prove that there exist integers $x_1, \dots, x_{2m}$ not all equal to 0 but also satisfying the inequality $|x_i| \le m$ for each $i = 1, \dots, m$ such that

$$\sum_{j=1}^{2m} a_{i,j}x_j = 0 \text{ for each } i = 1, \dots, m.$$

Solution. We consider the set S consisting of all tuples $(b_1, \dots, b_m)$ with the property that there exist some $\epsilon_1, \dots, \epsilon_{2m} \in \{0, 1, \dots, m\}$ such that for each $i = 1, \dots, m$, we have

$$b_i = \sum_{j=1}^{m} a_{i,j}\epsilon_j.$$

Clearly, each b_i satisfies the inequality $|b_i| \le m^2$ since $a_{i,j} \in \{-1, 0, 1\}$. So, the set S contains at most $(m^2+1)^m$ distinct tuples. On the other hand, we have exactly $(m+1)^{2m} > (m^2+1)^m$ elements, thus showing that there must be two tuples $(\epsilon_1, \dots, \epsilon_{2m})$ and $(\delta_1, \dots, \delta_{2m})$ (where each ϵ_j and each δ_j is an integer between 0 and m) such that for each $i = 1, \dots, m$, we have

$$\sum_{j=1}^{2m} a_{i,j}\epsilon_j = \sum_{j=1}^{2m} a_{i,j}\delta_j.$$

Therefore, letting $x_j = \epsilon_j - \delta_j$ for each $j = 1, \dots, 2m$ yields integer numbers satisfying the inequality $|x_j| \le m$ for which we have that

$$\sum_{j=1}^{2m} a_{i,j}x_j = 0 \text{ for each } i = 1, \dots, m.$$