PUTNAM PRACTICE SET 4

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Problem 1. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be defined as follows: for each positive integer n, we let $f(n)$ be the sum of the digits of n. Find

$$f\left(f\left(f\left(2018^{2018}\right)\right)\right).$$

Solution. We have that

$$2018^{2018} < 3^{2018} \cdot 10^{3\cdot 2018} < 10^{1009} \cdot 10^{6054} = 10^{7063};$$

so, 2018^{2018} has at most 7063 digits and therefore,

$$f\left(2018^{2018}\right) < 9 \cdot 7063 < 10^5.$$

So, $f(n)$ has at most 5 digits thus proving that

$$f\left(f\left(2018^{2018}\right)\right) < 9 \cdot 5 = 45.$$

Furthermore, $f(n) \equiv n \pmod 9$; so, since

$$2018^{2018} \equiv 2^{2018} \equiv 2^{6\cdot 336+2} \equiv 2^2 \equiv 4 \pmod 9,$$

we conclude that

$$f\left(f\left(2018^{2018}\right)\right) \in \{4, 13, 22, 31, 40\}.$$

This means that $f\left(f\left(f\left(2018^{2018}\right)\right)\right) = 4$.

Problem 2. Let $x, y \in \mathbb{R}$ such that $x + y = 1$. Prove that

$$x^{m+1} \cdot \left(\sum_{j=0}^{n} \binom{m+j}{j} y^j\right) + y^{n+1} \cdot \left(\sum_{i=0}^{m} \binom{n+i}{i} x^i\right) = 1,$$

for each $m, n \in \mathbb{N}$.

Solution. We have, from the generalized binomial expansion for all x satisfying $|x| < 1$ that

$$(1-x)^{-n-1}$$

$$= \sum_{i=0}^{\infty} \frac{(-n-1)(-n-2)\cdots(-n-i)}{i!} \cdot (-x)^i$$

$$= \sum_{i=0}^{\infty} \binom{n+i}{i} x^i.$$

So, the function $h_{m,n}(x) := (1-x)^{n+1} \cdot \left(\sum_{i=0}^{m} \binom{n+i}{i} x^i\right)$ is an analytic function for all $|x| < 1$ and moreover, it is of the form $1 - x^{m+1} \cdot g_{m,n}(x)$, where $g_{m,n}(x)$ is another analytic function. So, the polynomial

$$P(x) := x^{m+1} \cdot \left(\sum_{j=0}^{n} \binom{m+j}{j} (1-x)^j\right) + (1-x)^{n+1} \cdot \left(\sum_{i=0}^{m} \binom{n+i}{i} x^i\right) - 1$$

is divisible by x^{m+1}. Similarly, arguing this time for the polynomial

$$Q(y) := (1-y)^{m+1} \cdot \left(\sum_{j=0}^{n} \binom{m+j}{j} y^j\right) + y^{n+1} \cdot \left(\sum_{i=0}^{m} \binom{n+i}{i} (1-y)^i\right) - 1,$$

we get that $y^{n+1} \mid Q(y)$. So, in other words, the polynomial $P(x)$ has 0 as a root of multiplicity at least $m+1$ and has 1 as a root of multiplicity at least $n+1$. Since $\deg(P) \le m+n+1$, we conclude that $P(x)$ is identically equal to 0, as desired.

Problem 3. For any real numbers $a < b$ and for any continuous function $g : [a,b] \longrightarrow \mathbb{R}$, we denote by $G(g)$ the graph of $g(x)$, i.e., the set

$$G(g) := \{(x,y)\colon a \le x \le b \text{ and } y = g(x)\}.$$

Also, for any function $f : [a,b] \longrightarrow \mathbb{R}$ and for any $c \in \mathbb{R}$, we denote by f_c the function $[a+c, b+c] \longrightarrow \mathbb{R}$ given by $f_c(x) := f(x-c)$. Find with proof all the real numbers $c \in (0,1)$ with the property that there exists some continuous function $f : [0,1] \longrightarrow \mathbb{R}$ (depending on c) such that:

- $f(0) = f(1) = 0$; and
- $G(f)$ and $G(f_c)$ are disjoint.

Solution. First we show that for any $n \in \mathbb{N}$, there is *no* continuous function $f^{(n)}$ having the desired properties for $c = \frac{1}{n}$, i.e., $f^{(n)}(0) = f^{(n)}(1) = 0$ and also $G\left(f^{(n)}\right) \cap G\left(f^{(n)}_{\frac{1}{n}}\right) = \emptyset$. Indeed, if there were such a continuous function $f^{(n)}$, then we note that $f^{(n)}\left(\frac{1}{n}\right) \neq 0$; so, without loss of generality, we may assume $f^{(n)}\left(\frac{1}{n}\right) > 0$. Then, we let $h_n : \left[\frac{1}{n}, 1\right] \longrightarrow \mathbb{R}$ be the function

$$h_n(x) := f^{(n)}(x) - f^{(n)}_{\frac{1}{n}}(x) = f^{(n)}(x) - f^{(n)}\left(x - \frac{1}{n}\right).$$

Clearly, h_n is a continuous function and $h_n\left(\frac{1}{n}\right) = f^{(n)}\left(\frac{1}{n}\right) - f^{(n)}(0) > 0$. Also, since $G\left(f^{(n)}\right) \cap G\left(f^{(n)}_{\frac{1}{n}}\right) = \emptyset$, we must have that $h_n(x) \neq 0$ for all $x \in \left[\frac{1}{n}, 1\right]$ and because h_n is continuous (while $h_n\left(\frac{1}{n}\right) > 0$), we get that $h_n(x) > 0$ for *all* $x \in \left[\frac{1}{n}, 1\right]$. So, $f^{(n)}\left(x - \frac{1}{n}\right) < f^{(n)}(x)$ for each $x \in \left[\frac{1}{n}, 1\right]$ and so,

$$0 = f^{(n)}(1) > f^{(n)}\left(\frac{n-1}{n}\right) > f^{(n)}\left(\frac{n-2}{n}\right) > \cdots > f^{(n)}(0) = 0,$$

which is a contradiction. So, indeed, no c of the form $\frac{1}{n}$ works. On the other hand, we can show that any other real number c in $(0,1)$ (which is not of the form $\frac{1}{n}$) would work, i.e., there would exist some continuous function $f^{(c)}$ with the properties that $f^{(c)}(0) = f^{(c)}(1) = 0$ and moreover, $G\left(f^{(c)}\right) \cap G\left(f^{(c)}_c\right) = \emptyset$.

We let $n \in \mathbb{N}$ with the property that $nc < 1 < (n+1)c$; according to our hypothesis on c, such a positive integer n is uniquely determined. We let $r := 1-nc$; this is a real number in $(0,c) \subset (0,1)$. We construct the function $f^{(c)}$ as a piecewise linear function on the intervals $[kc, kc+r]$ and respectively $[kc+r,(k+1)c]$ for each $k = 0,\dots,n$, respectively for each $k = 0,\dots,n-1$. So, the function is piecewise linear and we have that $f^{(c)}(kc) = k$ for each $k = 0,\dots,n$, while $f^{(c)}(kc+r) = -(n-k)$ for each $k = 0,\dots,n$. Now, we claim that $f^{(c)}(x) \neq f_c^{(c)}(x)$ for each $x \in [c,1]$, i.e., $f^{(c)}(x) \neq f^{(c)}(x-c)$ for each $x \in [c,1]$. Indeed, if such a real number x would exist, then we let k be the unique integer in the set $\{1,\dots,n\}$ such that $x \in [kc,(k+1)c)$. Furthermore, we have two possibilities: either $x \in [kc, kc+r)$ or $x \in [kc+r,(k+1)c)$.

Case 1. If $x \in [kc, kc+r)$, then $f^{(c)}(x) = k - (x-kc)\cdot \frac{n}{r}$, while $x - c \in [(k-1)c,(k-1)c+r)$ and so, $f^{(c)}(x-c) = k-1-(x-kc)\cdot\frac{n}{r}$ and so, $f^{(c)}(x) \neq f^{(c)}(x-c)$.

Case 2. If $x \in [kc+r,(k+1)c)$, then $f^{(c)}(x) = k-n+(x-kc-r)\cdot\frac{n+1}{c-r}$, while $x-c \in [(k-1)c+r, kc)$ and so, $f^{(c)}(x-c) = k-1-n+(x-kc-r)\cdot\frac{n+1}{c-r}$ thus showing that $f^{(c)}(x) \neq f^{(c)}(x-c)$, as desired.

Problem 4. Find all polynomials $P \in \mathbb{R}[x,y]$ satisfying the following properties:

(1) there exists some $n \in \mathbb{N}$ with the property that $P(tx,ty) = t^n P(x,y)$ for all $t,x,y \in \mathbb{R}$;
(2) $P(a+b,c) + P(b+c,a) + P(c+a,b) = 0$ for all $a,b,c \in \mathbb{R}$; and
(3) $P(1,0) = 1$.

Solution. The first condition tells us that $P(x,y)$ is a homogenous polynomial in 2 variables; so, it can be written as $y^n \cdot R(x/y)$. However, it's more suitable to write

$$P(x,y) = (x+y)^n \cdot P\left(\frac{x}{x+y}, \frac{y}{x+y}\right) = (x+y)^n \cdot Q\left(\frac{y}{x+y}\right),$$

where $Q(z) := P(1-z,z)$ is a polynomial of one variable. Then condition (2) yields

$$Q\left(\frac{c}{a+b+c}\right) + Q\left(\frac{a}{a+b+c}\right) + Q\left(\frac{b}{a+b+c}\right) = 0 \text{ as long as } a+b+c \neq 0.$$

So, this means that for any $u,v \in \mathbb{R}$, we have

$$Q(u) + Q(v) + Q(1-u-v) = 0.$$

The above condition yields that Q must be a linear polynomial and furthermore, $Q(t) = 3ct - c$ for some $c \in \mathbb{R}$. Hence, $P(x,y) = c(x+y)^{n-1}(2y-x)$.

Problem 5. Let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. Prove that there exist infinitely many $n \in \mathbb{N}$ for which there exist $(k,m,x,y) \in \mathbb{N}^4$ such that $a_n = xa_k + ya_m$.

Solution. Actually, given any positive integer M, since there exist finitely many residue classes modulo M, there must exists an infinite subset $S \subseteq \mathbb{N}$ such that for each $n,k \in S$, we have that $a_n \equiv a_k \pmod{M}$ and therefore, letting M be any given a_m, if $n > k$ then there exists some $y \in \mathbb{N}$ such that $a_n = a_k + ya_m$.

Problem 6. Let $0 < x_1 < \cdots < x_n < \frac{\pi}{2}$ be real numbers. Prove that:

$$\sum_{i=1}^{n-1} \sin(2x_i) - \sum_{i=1}^{n-1} \sin(x_i - x_{i+1}) < \frac{\pi}{2} + \sum_{i=1}^{n-1} \sin(x_i + x_{i+1}).$$

Solution. Using trigonometric identities, we are left too prove the following inequality:

$$\begin{aligned} &\frac{\pi}{2} \\ &> \sum_{i=1}^{n-1} \left(\sin(2x_i) - \sin(x_i - x_{i+1}) - \sin(x_i + x_{i+1})\right) \\ &= \sum_{i=1}^{n-1} \left(2\sin(x_i)\cos(x_i) - 2\sin(x_i)\cos(x_{i+1})\right) \\ &= 2 \cdot \sum_{i=1}^{n-1} \sin(x_i) \cdot \left(\cos(x_i) - \cos(x_{i+1})\right). \end{aligned}$$

However, for each $i = 1, \ldots, n-1$, $\sin(x_i) \cdot (\cos(x_i) - \cos(x_{i+1}))$ represents the area of a rectangle of height $\sin(x_i)$ contained in the upper right quadrant of the unit circle (with one side lying on the x-axis). Since all these rectangles have disjoint interiors and they're all contained in a quadrant of area $\pi/4$, we obtain the desired conclusion.