PUTNAM PRACTICE SET 31: SOLUTIONS

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Problem 1. Is there an infinite sequence of real numbers $\{a_n\}_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} a_n^m = m$$

for each $m \in \mathbb{N}$?

Solution. No; here's why. Assuming there exists such an infinite sequence, then there are two possibilities:

Case 1. For each $n \geq 1$, we have that $0 \leq a_n^2 \leq 1$. In this case we would have that $a_n^4 \leq a_n^2$ for each $n \geq 1$, which contradicts the fact that

$$\sum_{n=1}^{\infty} a_n^4 = 4 > 2 = \sum_{n=1}^{\infty} a_n^2.$$

Case 2. There exists $k \geq 1$ such that $a_k^2 > 1$. But then for surfficiently large m, we would have that

$$a_k^{2m} > 2m$$

since the function $x \mapsto (a_k^2)^x - x$ tends to infinity as x tends to infinity (because $a_k^2 > 1$). So, we get a contradiction in both cases, thus showing that indeed no such sequence $\{a_n\}_{n\geq 1}$ exists.

Problem 2. Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be placed into k boxes with the sum of the integers in each box being the same across all boxes?

Solution. The largest such number k is $\left[\frac{n+1}{2}\right]$. Indeed, if $n = 2\ell$, we can simply partition $\{1, 2, \dots, 2\ell\}$ into ℓ subsets with 2 elements each:

$$\{1, 2\ell\};\ \{2, 2\ell - 1\};\ \cdots;\ \{\ell, \ell + 1\},$$

while if $n = 2\ell + 1$, then we partition $\{1, 2, \dots, 2\ell + 1\}$ into $\ell + 1$ subsets:

$$\{2\ell + 1\};\ \{1, 2\ell\};\ \{2, 2\ell - 1\};\ \cdots;\ \{\ell, \ell + 1\}.$$

So, $k = \left[\frac{n+1}{2}\right]$ is definitely a possibility as desired; now, the point is to prove that we cannot use a partition in *more* than $\left[\frac{n+1}{2}\right]$ subsets of $\{1, 2, \dots, n\}$ such that the sum of numbers in each subset is the same. To see that this is not possible, we note that n must be contained in one of the subsets and so, the common sum for the integers in each subset is at least n and thus the number k of subsets in our partition must satisfy the inequality:

$$kn \leq 1 + 2 + \cdots + n$$

and thus, $k \leq \frac{n+1}{2}$, as desired.

Alternatively, we could have noticed that using more than $\left[\frac{n+1}{2}\right]$ *subsets for our partition yields that we have at* **least** *two subsets in our partition with only one* *element in them, and this would be impossible since then not all subsets would contain integers adding up to the same number.*

Problem 3. Find all differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for each $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$.

Solution. Clearly, any linear function $f(x) = ax + b$ satisfies the condition from this problem; we'll prove next that the linear functions are the only functions meeting the given conditions for all $n \in \mathbb{N}$ (and all $x \in \mathbb{R}$).

Now, using the hypothesis for $n = 1$, we get

$$f'(x) = f(x+1) - f(x). \tag{1}$$

On the other hand, using the hypothesis for $n = 2$, we get

$$2f'(x) = f(x+2) - f(x)$$

and so, combining the last equality with (1), which yields also that $f'(x+1) = f(x+2) - f(x+1)$, we obtain

$$2f'(x) = f(x+2)-f(x) = (f(x+2) - f(x+1))+(f(x+1) - f(x)) = f'(x+1)+f'(x)$$

and so, $f'(x+1) = f'(x)$ for all $x \in \mathbb{R}$. We consider the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$g(x) = f(x+1) - f(x),$$

for which we have that

$$g'(x) = f'(x+1) - f'(x) = 0.$$

So, there exists a positive real number a such that

$$g(x) = a \text{ for all } x \in \mathbb{R}.$$

In particular, because of (1) (and that $g(x) = f(x+1) - f(x)$), we conclude that

$$f'(x) = a \text{ for all } x \in \mathbb{R}.$$

In conclusion, since the derivative of f is constant, we conclude that indeed, $f(x) = ax + b$ must be a linear function (for given $a, b \in \mathbb{R}$).

Problem 4. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous partial derivatives, which satisfies the following equation:

$$f(x,y) = a \cdot \frac{\mathrm{df}}{\mathrm{dx}}(x,y) + b \cdot \frac{\mathrm{df}}{\mathrm{dy}}(x,y)$$

for each $(x, y) \in \mathbb{R}^2$. Prove that if there exists a constant M such that

$$|f(x,y)| \le M \text{ for each } (x,y) \in \mathbb{R}^2,$$

then f must be identically equal to 0.

Solution. Now, if $a = b = 0$, then the conclusion is trivial. So, from now on, we assume $(a, b) \neq (0, 0)$.

We let $(x_0, y_0) \in \mathbb{R}^2$ and we will show that $f(x_0, y_0) = 0$.

We consider the function $h : [0, 1] \longrightarrow \mathbb{R}^2$ given by

$$h(t) = (x_0 + at, y_0 + bt).$$

Then our hypothesis yields that for the function $g = f \circ h : [0, 1] \longrightarrow \mathbb{R}$, we have:

$$g'(t) = g(t),$$

thus showing that there exists some constant $C_0 \in \mathbb{R}$ such that $g(t) = C_0 e^t$. On the other hand, we know that $|f|$ is bounded above, which means that $|g|$ is bounded above; this can only happen if $C_0 = 0$, which automatically yields that

$$g(0) = f(x_0, y_0) = 0,$$

as desired.