PUTNAM PRACTICE SET 30: SOLUTIONS

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Problem 1. Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be a continuous function on the unit square such that the partial derivatives df/dx and df/dy exists and are continuous on the interior $(0,1)^2$. Prove or disprove whether there always exists some point $(x_0, y_0) \in (0,1)^2$ such that:

$$\frac{\mathrm{df}}{\mathrm{dx}}(x_0,y_0) = \int_0^1 f(1,y)\mathrm{dy} - \int_0^1 f(0,y)\mathrm{dy} \text{ and } \frac{\mathrm{df}}{\mathrm{dy}}(x_0,y_0) = \int_0^1 f(x,1)\mathrm{dx} - \int_0^1 f(x,0)\mathrm{dx}$$

Solution. We show that the statement doesn't always hold; a counterexample is provided by the function $f(x,y) = x\sin(2\pi y)$. In this case, we have that

$$\int_0^1 f(0,y)\mathrm{dy} = \int_0^1 f(1,y)\mathrm{dy} = \int_0^1 f(x,0)\mathrm{dx} = \int_0^1 f(x,1)\mathrm{dx} = 0.$$

On the other hand,

$$\frac{\mathrm{df}}{\mathrm{dx}}(x,y) = \sin(2\pi y) \text{ and } \frac{\mathrm{df}}{\mathrm{dy}}(x,y) = 2\pi x\cos(2\pi y)$$

and so, both derivatives being 0 at some point $(x_0,y_0) \in (0,1)^2$ would force first that $y_0 = \frac{1}{2}$, but then x_0 would need to be equal to 0, which means that the point (x_0,y_0) would not be *inside* the unit square.

Problem 2. Show that every positive rational number can be written as a quotient of factorials of primes (not necessarily distinct); for example,

$$\frac{6}{7} = \frac{3! \cdot 3! \cdot 5!}{7!}.$$

Solution. We write a given positive rational number in its lowest terms as $\frac{a}{b}$ with $a,b \in \mathbb{N}$ and $\gcd(a,b) = 1$. We prove our statement by induction on the largest prime p dividing either a or b. The first case can be taken to be even the case when there is no prime dividing either a or b, i.e., $a = b = 1$, in which case, clearly

$$\frac{1}{1} = \frac{2!}{2!}, \text{ for example.}$$

On the other hand, even the case $p = 2$ is the largest prime dividing either a or b follows just as easily since this would mean that $\frac{a}{b} = 2^\ell$ for some $\ell \in \mathbb{Z}$ and so, we could simply write

$$\frac{a}{b} = (2!)^\ell.$$

Now, we assume that any fraction $\frac{a}{b}$ in which the largest prime dividing a or b is less than a given prime number p can be written in the form indicated in our conclusion. So, we assume p^k is the largest power of p appearing in $\frac{a}{b}$ (so, in particular, we allow for the possibility that k is negative, which corresponds to the case when $p \mid b$). But then, we have

$$\frac{a}{b} = (p!)^k \cdot \frac{c}{d \cdot ((p-1)!)^k},$$

for some positive integers c and d. **Very important:** both c and d are divisible by primes less than p; also, $(p-1)!$ is divisible by primes less than p and therefore, the inductive hypothesis can be applied and so, $\frac{c}{d \cdot ((p-1)!)^k}$ is indeed written as desired (and in turn $\frac{a}{b}$ is written as in the conclusion for this problem).

Problem 3. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c, can one travel from 0 to 1 in a finite number of jumps with total cost equal to c?

Solution. We are asked the following: find all possible values for the real number c for which there exists $n \in \mathbb{N}$ and real numbers:

$$0 = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = 1$$

such that

$$\sum_{i=1}^{n} a_i^2 \cdot (a_i - a_{i-1}) = c.$$

Now, on one hand, using right Riemann sums, we immediately see that

$$\sum_{i=1}^{n} a_i^2 \cdot (a_i - a_{i-1}) > \int_0^1 x^2 \mathrm{dx} = \frac{1}{3}.$$

Also, we clearly have that

$$\sum_{i=1}^{n} a_i^2 \cdot (a_i - a_{i-1}) \le \sum_{i=1}^{n} (a_i - a_{i-1}) = 1,$$

with equality in the case $n = 1$ and so, $a_0 = 0 < 1 = a_1$. So, the numbers c as above must be contained in the interval $(1/3, 1]$. We've just seen that $c = 1$ is possible; we'll prove next that *each* real number $c \in (1/3, 1)$ can also be attained.

First, we see that each number c arbitrarily closer to $\frac{1}{3}$, but larger than $\frac{1}{3}$ can also be achieved. Indeed, let

$$a_i = \frac{i}{n} \text{ for } i = 0, 1, \ldots, n.$$

Then

$$\sum_{i=1}^{n} a_i^2 (a_i - a_{i-1}) = \frac{1}{n} \cdot \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 = \frac{(n+1)(2n+1)}{6n^2},$$

which clearly converges to $\frac{1}{3}$ (from above) as $n \to \infty$. **(Actually, we can see this last fact also by interpreting once again the above sum as a Riemann sum for the integral of x^2 over the interval $[0, 1]$ and see that when n tends to infinity, the Riemann sum approaches the actual value of the integral, which is $\frac{1}{3}$.)**

Now, for the same choice of $a_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$ (for some given positive integer n), we let for each $k = 1, \ldots, n$:

$$c_k := a_k^3 + \sum_{i=k+1}^{n} a_i^2 \cdot (a_i - a_{i-1}).$$

So, $c_1 = \frac{(n+1)(2n+1)}{6n^2}$ and $c_n = 1$. We claim that each number c from c_1 to c_n can also be achieved by some suitable sequence. Indeed, for each $k = 2, \ldots, n$, we consider the sequence

$$0, x, a_k, a_{k+1}, \ldots, a_n$$

where we let x vary between 0 and a_{k-1}. We compute the cost associated to the above sequence, as a function of x:

$$c(x) := x^3 + a_k^2(a_k - x) + \sum_{i=k+1}^{n} a_i^2(a_i - a_{i-1})$$

which can be written in terms of c_k as follows:

$$c(x) = x^3 - xa_k^2 + c_k = c_k - x(a_k^2 - x^2).$$

Now, clearly, the function

$$x \mapsto c_k - x(a_k^2 - x^2)$$

for $x \in [0, a_{k-1}]$ varies continuously between c_{k-1} (attained when $x = a_{k-1}$) and c_k (attained when $x = 0$). So, all the values between c_{k-1} and c_k are taken for each $k = 2, \ldots, n$; therefore, indeed all values $c \in (1/3, 1]$ can be achieved.

Problem 4. Say that a polynomial $P \in \mathbb{R}[x, y]$ is balanced if the average value of the polynomial on each circle centered at the origin is 0, i.e.,

$$\int_C P(x, y) = 0$$

for any circle C in the cartesian plane. The balanced polynomials of degree 2021 form an $\mathbb{R}$-vector space V; find $\dim_{\mathbb{R}} V$.

Solution. Each polynomial $P \in \mathbb{R}[x, y]$ of degree d can be written as sum of homogeneous polynomials P_i of degrees i, for $i = 0, 1, \ldots, d$. On the other hand, for any given homogeneous polynomial Q and any circle C of radius r centered at the origin, we have that

$$\int_C Q(x, y) = r^i \cdot \int_{C_1} Q(x, y),$$

where C_1 is the unit circle centered at the origin. So, letting

$$A_i := \int_{C_1} P_i(x, y) \text{ for } i = 0, \ldots, d,$$

we get that

$$\int_C P(x, y) = 0 \text{ for each circle } C \text{ centered at the origin}$$

if and only if

$$\sum_{i=0}^{d} A_i r^i = 0 \text{ for any } r > 0 \tag{1}$$

because for a circle C of radius r centered at the origin, we have

$$\int_C P_i(x,y) = \int_0^{2\pi} P_i(r\cos(t), r\sin(t))\mathrm{dt} = r^i \cdot \int_0^{2\pi} P_i(\cos(t),\sin(t))\mathrm{dt} = r^i \cdot \int_{C_1} P_i(x,y).$$

Clearly, the above equation (1) is a polynomial identity (because the A_i's are simply constants); so, (1) holds if and only if

$$A_i = 0 \text{ for } i = 0, 1, \dots, d.$$

Now, for each odd integer i (from 0 to d), we have that $A_i = 0$ since for any odd homogeneous polynomial $Q \in \mathbb{R}[x,y]$ (i.e., when $Q(-x,-y) = -Q(x,y)$), we have that

$$\int_{C_1} Q(x,y) = 0. \tag{2}$$

We can see the validity of (2) from the fact that

$$\int_{C_1} Q(x,y) = \int_0^{2\pi} Q(\cos(t),\sin(t))\mathrm{dt} = \int_0^{2\pi} Q(\cos(t+\pi),\sin(t+\pi))\mathrm{dt} =$$
$$= \int_0^{2\pi} Q(-\cos(t),-\sin(t))\mathrm{dt} = -\int_0^{2\pi} Q(\cos(t),\sin(t))\mathrm{dt}.$$

So, $A_i = 0$ whenever i is odd regardless of the homogeneous polynomials P_i. Now, when i is even, the condition $A_i = 0$ imposes some linear condition on the polynomial P_i (the coefficients of this linear condition are given by integrating $x^j y^{i-j}$ over C_1 for $j = 0, \dots, i$), and thus, in turn, it imposes a linear condition on the coefficients of P. **(Also, note that these conditions are independent since they refer to different coefficients because each P_i is homogeneous of degree i.)**

Now, for i even, the corresponding linear condition is nontrivial since whenever $0 \le j \le i$ is also even, then integrating $x^j y^{i-j}$ over C_1 would always give us a strictly positive real number. Thus, we have for each even $0 \le i \le d$ some nontrivial linear relation among the coefficients of $x^j y^{i-j}$ of $P(x,y)$ (for $0 \le j \le i$). Hence, for an odd integer d (such as 2021), we have no restriction for the coefficients of the odd (total degree) monomials in $P(x,y)$, and we have $\frac{d+1}{2}$ linearly independent relations to be satisfied by the coefficients of the monomials of even (total degree) in $P(x,y)$. Therefore, the dimension of our given linear space of polynomials must be

$$-\frac{d+1}{2} + \sum_{i=0}^{d}(i+1) = \frac{(d+1)^2}{2}.$$