PUTNAM PRACTICE SET 3

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Problem 1. The set A has 6 elements. Determine with proof whether we can find distinct subsets $B_1, \dots, B_m$ of A (for some $m \in \mathbb{N}$) satisfying the following properties:

- each B_i has exactly 3 elements;
- for any two elements x and y of A, there exist precisely two indices $1 \le i < j \le m$ such that $x, y \in B_i$ and $x, y \in B_j$.

Solution. First, we note that $m = 10$ since each set B_i yields 3 distinct pairs and then the above condition asks that each pair (i, j) of elements of A (where we consider $A = \{1, 2, 3, 4, 5, 6\}$) must appear exactly twice; so,

$$3 \cdot m = 2 \cdot \binom{6}{2} \text{ yields } m = 10.$$

Claim. We cannot have three sets B_i be of the form $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4\}$ and $\{a_1, a_3, a_4\}$.

Proof of Claim. Without loss of generality, let's assume $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$ and $B_3 = \{1, 3, 4\}$. But then we need two sets B_i to contain $\{1, 5\}$; however none of these B_i would be allowed to contain either 2, 3 or 4, which would be a contradiction.

So, assume now that $B_1 = \{1, 2, 3\}$ and $B_2 = \{1, 2, 4\}$ (which we can always assume after relabelling our elements). Also, let B_3 contain $\{1, 3\}$; then we know that $B_3 \neq \{1, 3, 4\}$ and therefore (again after a relabelling of our elements), we may assume $B_3 = \{1, 3, 5\}$. Next, we assume B_4 and B_5 contain $\{1, 6\}$; then neither one can contain 2 or 3 and therefore, it must contain 4, respectively 5. So, assume that $B_4 = \{1, 4, 6\}$ and $B_5 = \{1, 5, 6\}$. Next we see that if B_6 contains $\{2, 3\}$ then its third element must be 6 and similarly, if B_7 contains $\{2, 4\}$ then its third element must be 5. Next assume B_8 and B_9 contain $\{3, 4\}$; we see that their third element cannot be 1 or 2, so they must be 5 and 6, i.e., $B_8 = \{3, 4, 5\}$ and $B_9 = \{3, 4, 6\}$. This leaves us with $B_{10} = \{2, 5, 6\}$. So, our sets are

$$\{1, 2, 3\},\ \{1, 2, 4\},\ \{1, 3, 5\},\ \{1, 4, 6\},\ \{1, 5, 6\}$$

$$\{2, 3, 6\},\ \{2, 4, 5\},\ \{3, 4, 5\},\ \{3, 4, 6\},\ \{2, 5, 6\}.$$

Problem 2. We consider the following real numbers:

$$x_1 \le x_2 \le \cdots \le x_n \text{ and } y_1 \le y_2 \le \cdots \le y_n.$$

Let $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ be any permutation. Prove that

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} \left(x_i - y_{\sigma(i)}\right)^2.$$

Solution. For any $1 \le i < j \le n$, we note that

$$\begin{aligned}(x_i - y_i)^2 + (x_j - y_j)^2 - (x_i - y_j)^2 - (x_j - y_i)^2 \\ = 2(x_i y_j + x_j y_i - x_i y_i - x_j y_j) \\ = -2(x_i - x_j)(y_i - y_j) \\ \le 0.\end{aligned}$$

So, if $\sigma(i) > \sigma(j)$ for some $i < j$, then switching $\sigma(i)$ and $\sigma(j)$ (i.e., composing σ with the transposition (i, j)) would only decrease (or leave unchanged) the sum $\sum_{i=1}^{n} \left(x_i - y_{\sigma(i)}\right)^2$. After applying finitely many transpositions, we're back at the identity permutation and at each step we only decreased (or perhaps left unchanged) the given sum.

Problem 3. For any given positive real numbers a, b, c, d, we let

$$S_{a,b,c,d} := \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

Find all possible values that are taken by $S_{a,b,c,d}$ as we vary a, b, c, d in the set of all positive real numbers.

Solution. Let $\mathbb{R}_+ := (0, +\infty)$ be the set of all (strictly) positive real numbers. We consider the function

$$f : \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+ \text{ given by}$$

$$f(a, b, c, d) = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

Clearly,

$$f(a, b, c, d) < \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{c+d} + \frac{d}{c+d} = 2$$

and also,

$$f(a, b, c, d) > \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1.$$

So, the image of f is contained in the open interval $(1, 2)$. On the other hand, f is continuous and moreover, for any given $b, d \in \mathbb{R}_+$ we have that

$$\lim_{a,c\to 0} f(a, b, c, d) = \lim_{a\to 0} \frac{a}{a+b+d} + \lim_{c\to 0} \frac{c}{b+c+d} + \lim_{a,c\to 0} \frac{b}{a+b+c} + \lim_{a,c\to 0} \frac{d}{a+c+d} = 0+0+\frac{b}{b}+\frac{d}{d} = 2,$$

while for any given $c, d \in \mathbb{R}_+$, we have

$$\lim_{a,b\to 0} f(a, b, c, d) = \frac{c}{c+d} + \frac{d}{c+d} = 1.$$

Now, since $\mathbb{R}_+^4$ is a connected set (i.e., it cannot be written as a union of two disjoint open subsets; note that a product of connected spaces is connected), then its image under a continuus function must be connected and therefore, it must be precisely the open interval $(1, 2)$.

Problem 4. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying the following properties:

(i) $0 \le a_n \le 1$ for each $n \ge 1$; and
(ii) $a_n - 2a_{n+1} + a_{n+2} \ge 0$ for each $n \ge 1$.

Prove that for each $n \geq 1$, we have that $0 \leq (n+1)(a_n - a_{n+1}) \leq 2$ for each $n \geq 1$.

Solution. First we note that $b_n := a_n - a_{n+1}$ must be nonnegative for each $n \geq 1$. Indeed, if $b_{n_0} < 0$ for some $n_0 \in \mathbb{N}$, then condition (ii) yields that

$$b_{n_0} \geq b_{n_0+1} \geq b_{n_0+2} \geq \cdots$$

and so,

$$a_{n_0+\ell} \geq a_{n_0} - \ell b_{n_0} \text{ for each } \ell \geq 1$$

and so, if ℓ is sufficiently large, then condition (i) would no longer hold because we would actually get that $a_n \to +\infty$ as $\ell \to \infty$ (note that $b_{n_0} < 0$ by our assumption). So, indeed, we must have that $b_n \geq 0$ and so, $(n+1)(a_n - a_{n+1}) \geq 0$.

Now, assume there exists some $n_1 \in \mathbb{N}$ such that $b_{n_1} > \frac{2}{n_1+1}$. We have that

$$b_1 \geq b_2 \geq \cdots \geq b_{n_1}$$

and so,

$$a_1 - a_{n_1+1} = \sum_{i=1}^{n_1} b_i > \frac{2n_1}{n_1+1}$$

and since $2n_1 \geq n_1 + 1$, we conclude that we would have that $a_1 - a_{n_1+1} > 1$ contradicting our hypothesis that both a_1 and a_{n_1+1} are in the interval $[0, 1]$. So, it must be that $(n+1)(a_n - a_{n+1}) \leq 2$ for each $n \geq 1$.

Problem 5. Let M be the set of all positive integers which do not contain the digit 9 in their decimal expansion. Prove that $\sum_{x \in M} \frac{1}{x} < 80$.

Solution. Let $k \in \mathbb{N}$. Then among all the numbers with k digits, we have only 8 possibilities for their first digit, while for each of the other digits, there are 9 possibilities. So, we have $8 \cdot 9^{k-1}$ numbers with exactly k digits which do not contain the digit 9. Furthermore, each such number with k digits is at least equal to 10^{k-1}. So,

$$\sum_{x \in M} \frac{1}{x} < \sum_{k=1}^{\infty} \frac{8 \cdot 9^{k-1}}{10^{k-1}} = 8 \cdot \sum_{\ell=0}^{\infty} \left(\frac{9}{10}\right)^{\ell} = 8 \cdot \frac{1}{1 - \frac{9}{10}} = 80,$$

as claimed.

Problem 6.

(A) Show that the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

is not a rational number.

(B) Let a and b be integers larger than 1. We construct two sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ as follows:

$$a_1 = a \text{ and for all } n \geq 1, \text{ we have } a_{n+1} = na_n - 1$$

and

$$b_1 = b \text{ and for all } n \geq 1, \text{ we have } b_{n+1} = nb_n + 1.$$

Prove that there exist at most finitely many pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $a_m = b_n$.

Solution.

(A) We argue by contradiction and therefore assume the series is a rational number $\frac{a}{b}$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Since

$$\frac{5}{3} = 1 + \frac{1}{2} + \frac{1}{6} < \sum_{n=1}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2,$$

we get that $b \geq 4$. Now, we note that

$$\sum_{n=b+1}^{\infty} \frac{1}{n!} < \frac{1}{(b-1)!} \cdot \sum_{n=b}^{\infty} \frac{1}{n(n+1)} = \frac{1}{b!}.$$

On the other hand,

$$\frac{a}{b} - \sum_{n=1}^{b} \frac{1}{n!} = \frac{c}{b!} \geq \frac{1}{b!},$$

where $c \in \mathbb{N}$ since $b!$ is the least common multiple of all the denominators and furthermore, the above difference must actually equal the tail of the infinite series (and therefore, it must be positive). The above two inequalities contradict each other, which concludes our proof of part (A).

(B) We first find the general formula for a_n, respectively b_n. We have:

$$\frac{a_{n+1}}{n!} = \frac{a_n}{(n-1)!} - \frac{1}{n!}.$$

So,

$$\frac{a_{n+1}}{n!} = a - \sum_{k=1}^{n} \frac{1}{k!};$$

thus

$$a_n = (n-1)! \cdot \left(a - \sum_{k=1}^{n-1} \frac{1}{k!} \right).$$

A similar analysis yields

$$b_n = (n-1)! \cdot \left(b + \sum_{k=1}^{n-1} \frac{1}{k!} \right).$$

Since $a \geq 2$ and $\sum_{k=1}^{\infty} \frac{1}{k!} < 2$, we conclude that $\lim_{n\to\infty} a_n = \infty$. Similarly, $\lim_{n\to\infty} b_n = \infty$.

We let $e := \sum_{k=0}^{\infty} \frac{1}{k!}$. Assume there exist infinitely many pairs $(m_i, n_i) \in \mathbb{N} \times \mathbb{N}$ such that $a_{m_i} = b_{n_i}$. Because $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$ (in other words, the sequences are not repeating after a while), then both the first and the second entries in those pairs (m_i, n_i) grow arbitrarily large (i.e., $\lim_{i\to\infty} m_i = \infty$ and $\lim_{i\to\infty} n_i = \infty$). Therefore

$$\lim_{i\to\infty} \frac{a - \sum_{k=1}^{m_i-1} \frac{1}{k!}}{b + \sum_{k=1}^{n_i-1} \frac{1}{k!}} = \frac{a-e+1}{b+e-1}. \tag{1}$$

Furthermore, since $e \notin \mathbb{Q}$ (as proven in part (A)), then $\frac{a-e+1}{b+e-1} \notin \mathbb{Q}$. Since $a_{m_i} = b_{n_i}$, the formulas for the a_k's and for the b_k's coupled with limit (1) yield that

$$\lim_{i\to\infty} \frac{(n_i-1)!}{(m_i-1)!} = \frac{a-e+1}{b+e-1} \notin \mathbb{Q}.$$

In case it exists, there are three possibilities for the limit

$$\lim_{i\to\infty} \frac{(n_i - 1)!}{(m_i - 1)!} \tag{2}$$

either 0, or 1, or ∞ depending on whether $m_i > n_i$ for all i sufficiently large, or $m_i = n_i$ for all i sufficiently large, or $m_i < n_i$ for all i sufficiently large. (If *at least two* of the possibilities: $m_i > n_i$, or $m_i = n_i$, or $m_i < n_i$ occur infinitely often then the limit (2) does not exist.) In either case, the limit (2) cannot be $\frac{a-e+1}{b+e-1} \notin \mathbb{Q}$ which yields the desired conclusion.