PUTNAM PRACTICE SET 27: SOLUTIONS

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Problem 1. Players A and B play the following game: each player (starting with player A) take turns in writing a real number in one of the (still) empty entries of a 2020-by-2020 matrix. Player A wins if the determinant of the matrix at the end is nonzero, while player B wins if the determinant of the matrix is zero. Who has a winning strategy?

Solution. Player B has a winning strategy since each time when player A writes a number x on some row, then player B writes the number $-x$ on another empty entry from the same row. Since the dimension of the matrix is *even*, then we're always guaranteed that player B will have at least one empty entry on the row where player A just wrote a number. Also, this strategy ensures that at the end of the game, the sum of the entries on each row equals 0, thus showing that the matrix admits a nontrivial vector in its null space (i.e., the vector whose all entries are equal to 1); hence the matrix is not invertible and so, its determinant must be 0.

Problem 2. Let $n \in \mathbb{N}$. We start with a finite sequence

$$a_1, a_2, \ldots, a_n$$

of positive integers and then at each step, we perform the following operation: if for some indices $1 \le i < j \le n$, we have that a_i does not divide a_j, then we replace a_i by $\gcd(a_i, a_j)$ and also, replace a_j by $\operatorname{lcm}[a_i, a_j]$.

Prove that after finitely many steps we can no longer perform any new operation. Furthermore, show that the final sequence we obtain is the same one regardless of the order of the above operations that we performed.

Solution. First we prove that our process ends in finitely many steps; we achieve this by showing that the elements of our sequence eventually stop changing (i.e., we eventually arrive at the situation that for each $1 \le i < j \le n$, we have $a_i \mid a_j$). In order to do this, we first observe that in our process *never* the product of all elements $\prod_{i=1}^n a_i$ changes since for any two positive integers a and b, we have

$$a \cdot b = \gcd(a, b) \cdot \operatorname{lcm}[a, b].$$

Next we see that in our process, each element of any new sequence will be a divisor of

$$\operatorname{lcm}[a_1, a_2, \ldots, a_n].$$

Also, a_n can only increase in our process; so, this means that eventually, the last element of the sequence a_n has to stabilize and never changes again. But then this means that we are left with a sequence of $n-1$ elements for which we perform the described operation and so, a repeated argument as above yields that our process ends in finitely many steps. (Alternatively, we could have argued this final step using induction on n.)

A completely different and shorter proof for the fact that our process ends after finitely many steps is to note that the quantity

$$\prod_{i=1}^{n} a_i^{n+1-i}$$

decreases due to our process (because if $a \cdot b = A \cdot B$ with $A \le a, b \le B$ and $1 \le i < j \le n$, then $a^{n+1-i}b^{n+1-j} \ge A^{n+1-i}B^{n+1-j}$) and so, in finitely many steps, we must finish changing our sequence.

Now, we show that no matter in which order we performed the allowed changes in our sequence, at the end, the final sequence is the same. For this, we use the fact that for each prime p and for each $m \in \mathbb{N}$, the *number* of elements a_i from our sequence which are divisible by p^m is unchanged at each change performed in our sequence. Indeed, for any two positive integers a and b, if both are divisible by p^m then so is their greatest common divisor and their least common multiple; on the other hand, if none of the two integers is divisible by p^m, then neither is their greatest common divisor, nor is their least common multiple divisible by p^m. Finally, if exactly one of them is divisible by p^m, while the other integer is not divisible by p^m, then their least common multiple is divisible by p^m, while their greatest common divisor is not divisible by p^m.

So, using the above fact regarding the divisibility by p^m, then for each prime p we know precisely which are the elements in our sequence divisible by various powers of p because in our final sequence we have that $a_i \mid a_j$ whenever $i < j$. So, for a given prime p and for the largest power p^m dividing exactly $k \ge 1$ of the initial elements of our sequence, then we know that

$$a_n, a_{n-1}, \dots, a_{n-k+1}$$

are the only elements of our final sequence divisible by p^m. This fact alone allows us to construct uniquely the final sequence for our process because this identifies uniquely the prime factors of each element of the final sequence.

Problem 3. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if} \quad x \le e \\ xf(\ln(x)) & \text{if} \quad x > e \end{cases}$$

Determine whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

Solution. We have that our function has the following form. Given the sequence $\{a_n\}_{n \ge 0}$ for which

$$a_0 = 1 \text{ and } a_{n+1} = e^{a_n} \text{ for } n \ge 0,$$

we have that on the interval $[a_n, a_{n+1}]$, the function $f(x)$ equals

$$x \cdot \ln(x) \cdot \ln(\ln(x)) \cdot \dots \cdot \ln(\ln(\cdots \ln(x))),$$

where in the last iterated logarithmic function, we have n nested logarithms. In particular, when $n = 0$, then on the interval $[1, e]$ (since $a_1 = e$), we have that $f(x) = x$, while when $n = 1$, then on the interval $[e, e^e]$ (since $a_2 = e^e$), we have that $f(x) = x \ln(x)$; next, on the interval $\left[e^e, e^{e^e}\right]$, the function $f(x)$ equals $x \ln(x) \ln(\ln(x))$.

Now, due to the integral test, the series

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

is divergent precisely when the integral

$$\int_1^{\infty} \frac{1}{f(x)} \mathrm{dx}$$

diverges, because the function $x \mapsto \frac{1}{f(x)}$ is decreasing (note that each time when we introduce a new nested logarithmic function on a new interval $[a_n, a_{n+1}]$, the corresponding n-th nested logarithmic function is always at least equal to 1 when evaluated on $[a_n, a_{n+1}]$). For each $n \geq 0$, we compute

$$\int_{a_n}^{a_{n+1}} \frac{1}{f(x)} \mathrm{dx}$$

by performing the substitution $x = e^t$ and we get (due to the formula for $f(x)$) that

$$\int_{a_n}^{a_{n+1}} \frac{1}{f(x)} \mathrm{dx} = \int_{a_{n-1}}^{a_n} \frac{1}{f(x)} \mathrm{dx}.$$

Therefore, we have (after repeating the above process) that for each $n \geq 0$,

$$\int_{a_n}^{a_{n+1}} \frac{1}{f(x)} \mathrm{dx} = \int_{a_0}^{a_1} \frac{1}{f(x)} \mathrm{dx} = \int_1^e \frac{1}{x} \mathrm{dx} = 1.$$

Therefore, the integral $\int_1^{\infty} \frac{1}{f(x)} \mathrm{dx}$ diverges and so, also our original series diverges.

Problem 4. Prove that there exists a positive constant c with the property that in any finite nontrivial group G, there exists a suitable subset $S \subseteq G$ satisfying the following two properties:

- $|S| \leq c \log(|G|)$; and
- for each $x \in G$, there exist distinct elements $x_1, \dots, x_k \in S$ such that $x = x_1 \cdot x_2 \cdot \cdots \cdot x_k$.

Solution. We show how to construct a sequence S of elements of G of cardinality of the order of $\log(|G|)$ (i.e., bounded above by a positive constant *times* $\log |G|$) such that each element of G can be obtained as a product of some subsequence of S.

As a matter of notation, for any sequence S of elements in G, we let $\tilde{S}$ be the set of elements of G which can be obtained as a product of elements from a subsequence of S.

Now, we start with S containing the identity element of G and then at each step of our process, we do the following:

- **either** each element of G is of the form $g_1^{-1} \cdot g_2$ with $g_1, g_2 \in \tilde{S}$. If that's the case, then each element of G can be obtained as a product of a subsequence of elements in $S \cup S^{-1}$, where S^{-1} is the set consisting of all elements of the form g^{-1} for $g \in S$.
- **or** there exists an element $h \in G$ which cannot be written as $g_1^{-1} g_2$ for any $g_1, g_2 \in \tilde{S}$. If that's the case, then we enlarge our sequence S by inserting also the element g. We claim that when we do this we will generate at least twice as many elements with our new sequence $S \cup \{h\}$ than we generated

with our sequence S. Indeed, the point is that for each $g_1 \in \tilde{S}$, we have that $g_1h \notin \tilde{S}$, because otherwise we would get that $h = g_1^{-1}g_2$ with $g_1, g_2 \in \tilde{S}$, contradiction. So, indeed, $\tilde{S} \cdot h$ is disjoint from $\tilde{S}$, thus proving that

$$\widetilde{S \cup \{h\}}$$

contains at least twice as many elements as $\tilde{S}$ does.

Now, the above process shows that at each step the number of elements in $\tilde{S}$ at least doubles itself and so, starting with $|\tilde{S}| = 1$ in at most $\log_2(|G|)$ we arrive at a sequence S which together with S^{-1} would generate the entire G. So, the size of our generating sequence is at most

$$\frac{2}{\log(2)} \cdot \log(|G|).$$