PUTNAM PRACTICE SET 26: SOLUTIONS

PROF. DRAGOS GHIOCA

Problem 1. Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence given by

$$a_1 = 1 \text{ and } a_{n+1} = 3a_n + \left[\sqrt{5}\cdot a_n\right] \text{ for } n \geq 1.$$

Compute a_{2021}.

Solution. We see that

$$a_{n+1} - 3a_n = [\sqrt{5}\cdot a_n] < \sqrt{5}a_n < [\sqrt{5}a_n] + 1 = a_{n+1} - 3a_n + 1.$$

So,

$$(3+\sqrt{5})a_n - 1 < a_{n+1} < (3+\sqrt{5})a_n$$

and thus, after multiplying by $3-\sqrt{5}$,

$$4a_n - 3 + \sqrt{5} < (3-\sqrt{5})a_{n+1} < 4a_n,$$

which leads us to the following inequalities:

$$3a_{n+1} - 4a_n < \sqrt{5}a_{n+1} < 3a_{n+1} - 4a_n + 3 - \sqrt{5}.$$

Finally, since $0 < 3-\sqrt{5} < 1$ *and* the elements of our sequence are integers, we conclude that

$$[\sqrt{5}a_{n+1}] = 3a_{n+1} - 4a_n.$$

This means that

$$3a_n - 4a_{n-1} = [\sqrt{5}a_n] = a_{n+1} - 3a_n$$

for all $n \geq 2$. So, we have that our sequence satisfies the following linear recurrence relation:

$$a_{n+1} - 6a_n + 4a_{n-1} = 0,$$

which means that the characteristic roots of our linear recurrence sequence are

$$3+\sqrt{5} \text{ and } 3-\sqrt{5}$$

and since $a_1 = 1$ and $a_2 = 5$, we get that

$$a_n = \frac{\sqrt{5}+1}{8\sqrt{5}}\cdot(3+\sqrt{5})^n + \frac{\sqrt{5}-1}{8\sqrt{5}}\cdot(3-\sqrt{5})^n,$$

which happens to be precisely

$$a_n = 2^{n-2}\cdot F_{2n+1},$$

where $\{F_k\}_{k\geq 1}$ is the Fibonacci sequence given by

$$F_1 = F_2 = 1 \text{ and } F_{k+2} = F_{k+1} + F_k \text{ for } k \geq 1.$$

Problem 2. Let $n \in \mathbb{N}$. Find the number of pairs of polynomials $(P(x), Q(x)) \in \mathbb{R}[x]\times\mathbb{R}[x]$ satisfying the following two conditions:

- $\deg(P) > \deg(Q)$; and
- $P^2(x) + Q^2(x) = x^{2n} + 1$.

Solution. Since $\deg(P) > \deg(Q)$, then the square of the leading coefficient of $P(x)$ equals 1, which means that we have two possibilities: either the leading coefficient of $P(x)$ equals 1, or it equals -1. Now, if $(P(x), Q(x))$ is a solution, then also $(-P(x), Q(x))$ is a solution, which means that we may as well compute the number of pairs of polynomials $(P(x), Q(x))$ satisfying the given two condition and also for which $P(x)$ is a monic polynomial, and then simply double that number of pairs.

Also, we note that $\deg(P) = n$ since $\deg(P) > \deg(Q)$ and $\deg(P^2 + Q^2) = 2n$. So, noting that we assumed $P(x)$ is monic, then we notice the following factorization:

$$(P(x) + iQ(x)) \cdot (P(x) - iQ(x)) =$$

$$\prod_{j=1}^{n} \left(x + \cos\left(\frac{(2j+1)\pi}{2n}\right) + i \cdot \sin\left(\frac{(2j+1)\pi}{2n}\right)\right) \cdot \left(x + \cos\left(\frac{(2j+1)\pi}{2n}\right) - i \cdot \sin\left(\frac{(2j+1)\pi}{2n}\right)\right).$$

So, $P(x) + iQ(x)$ and $P(x) - iQ(x)$ are polynomials which are complex conjugates of each other and so, $P(x) + iQ(x)$ is a polynomial of degree n, which is a product of n parenthesis from the right hand side of the above factorization, where each parenthesis belongs to precisely one of the two possibilities

$$x + \cos\left(\frac{2\pi(2j+1)}{4n}\right) + i \cdot \sin\left(\frac{2\pi(2j+1)}{4n}\right) \textbf{ or } x + \cos\left(\frac{2\pi(2j+1)}{4n}\right) + i \cdot \sin\left(\frac{2\pi(2j+1)}{4n}\right)$$

for each $j = 1, \dots, n$. In conclusion, there are 2^n such possibilities for $P(x) + iQ(x)$ (and for each one of them, $P(x) - iQ(x)$ is uniquely determined as a product of the remaining n parenthesis from the left hand side of the above factorization), which means there are 2^n pairs of polynomials $(P(x), Q(X))$ such as above with $P(x)$ monic. (**Note that** $(P(x), Q(x))$ **is uniquely determined by the pair** $(P(x) + iQ(x), P(x) - iQ(x))$**.**) Therefore, overall, there are $2 \cdot 2^n = 2^{n+1}$ possible pairs of polynomials as desired.

Problem 3. Let $k \in \mathbb{N}$. Prove that there exist polynomials $P_0, P_1, \dots, P_{k-1}$ (which may depend on k) with the property that for each $n \in \mathbb{N}$, we have

$$\left[\frac{n}{k}\right]^k = P_0(n) + P_1(n) \cdot \left[\frac{n}{k}\right] + P_2(n) \cdot \left[\frac{n}{k}\right]^2 + \cdots + P_{k-1}(n) \cdot \left[\frac{n}{k}\right]^{k-1},$$

where (as always) $[x]$ is the integer part of the real number x.

Solution. For each integer n, we have that

$$\left[\frac{n}{k}\right] = \frac{n-j}{k},$$

for some integer $j \in \{0, 1, \dots, k-1\}$. So, this means that for each $n \in \mathbb{Z}$, we have that

$$\left(\left[\frac{n}{k}\right] - \frac{n}{k}\right) \cdot \left(\left[\frac{n}{k}\right] - \frac{n-1}{k}\right) \cdot \cdots \cdot \left(\left[\frac{n}{k}\right] - \frac{n-(k-1)}{k}\right) = 0.$$

So, simply expanding the above identity and grouping terms containing the same power of $\left[\frac{n}{k}\right]$ leads to the construction of the polynomials $P_0, P_1, \dots, P_{k-1}$ as desired.

Problem 4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with the property that

$$f(x, y) + f(y, z) + f(z, x) = 0,$$

for all real numbers x, y and z. Prove that there must exist another function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x, y) = g(x) - g(y),$$

for all real numbers x and y.

Solution. First we substitute: $x = y = z = 0$ in our functional identity and notice then that $f(0, 0) = 0$. Next we substitute $y = z = 0$ and obtain that

$$f(0, x) = -f(x, 0).$$

Finally, substituting $z = 0$ and letting x and y arbitrary, we obtain

$$f(x, y) + f(y, 0) + f(0, x) = 0, \text{ i.e.}$$

$$f(x, y) = -f(0, x) - f(y, 0) = f(x, 0) - f(y, 0),$$

thus proving that the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) := f(x, 0)$ satisfies the desired property for our conclusion.