PUTNAM PRACTICE SET 25: SOLUTIONS

PROF. DRAGOS GHIOCA

Problem 1. Let $n \in \mathbb{N}$ and let $a_1, \dots, a_n \in \mathbb{R}$. Show that there exists an integer m and some nonempty subset $S \subseteq \{1, \dots, n\}$ with the property that

$$\left| m + \sum_{i \in S} a_i \right| \le \frac{1}{n+1}.$$

Solution. We consider the fractional parts $\{\cdot\}$ of the following numbers:

$$s_k := \sum_{i=1}^{k} a_i \text{ for } k = 1, \dots, n.$$

Case 1. There exists $1 \le i < j \le n$ such that

$$|\{s_j\} - \{s_i\}| \le \frac{1}{n+1}.$$

In this case, writing $s_j = \{s_j\} + m_j$ and $s_i = \{s_i\} + m_i$ for some integers m_i and m_j (actually their respective integer parts $[\cdot]$), then we get:

$$|s_j - m_j - (s_i - m_i)| \le \frac{1}{n+1},$$

which means that

$$\left| \sum_{i < k \le j} a_k - (m_j - m_i) \right| \le \frac{1}{n+1}.$$

So, letting $m := m_i - m_j$, then we obtain the desired conclusion.

Case 2. For each $i \ne j$, we have that

$$|\{s_j\} - \{s_i\}| > \frac{1}{n+1}.$$

In this case, ordering the n fractional parts $\{s_k\}$ for $1 \le k \le n$, we see that they live in $[0, 1)$ and the distance between any two of them is greater than $\frac{1}{n+1}$, which means that:

- **either** $\{s_{i_0}\} \le \frac{1}{n+1}$, where $\{s_{i_0}\}$ is the smallest of the above fractional parts, in which case, the conclusion follows easily (we simply take $S = \{1, \dots, i_0\}$ and $m = -[s_{i_0}]$).
- **or** $1 - \{s_{j_0}\} < \frac{1}{n+1}$, where $\{s_{j_0}\}$ is the largest of the above fractional parts. In this case, we take $S = \{1, \dots, j_0\}$ and $m = -1 - [s_{j_0}]$ and still obtain the desired conclusion.

Problem 2. For each continuous function $f : [0, 1] \longrightarrow \mathbb{R}$, let

$$I(f) := \int_0^1 x^2 f(x) \mathrm{dx} - \int_0^1 x f(x)^2 \mathrm{dx}.$$

Find the maximum of $I(f)$ over all possible continuous functions f.

Solution. We compute

$$I(f)=\int_0^1\left(x^2f(x)-xf^2(x)\right)\mathrm{dx}=\int_0^1 x\cdot\left(xf(x)-f^2(x)\right)\mathrm{dx}=\int_0^1 x\cdot\left(-\frac{x^2}{4}+xf(x)-f^2(x)\right)+\frac{x^3}{4}\mathrm{dx}$$

and since

$$\frac{x^2}{4}-xf(x)+f^2(x)=\left(\frac{x}{2}-f(x)\right)^2\geq 0,$$

we see that

$$I(f)\leq\int_0^1\frac{x^3}{4}\mathrm{dx}=\frac{1}{16}.$$

The maximum $\frac{1}{16}$ is attained when $f(x)=\frac{x}{2}$ (which is a *continuous* function).

Problem 3. Let c be a real number greater than 1 and let $g\in\mathbb{R}[x]$ be a non-constant polynomial with the property that there exists an infinite sequence $\{k_n\}\subseteq\mathbb{N}$ with the property that for each $n\geq 1$, we have that there exists some $\ell_n\in\mathbb{N}$ with the property that

$$g\left(c^{k_n}\right)=c^{\ell_n}.$$

Find all such polynomials g.

Solution. Let $d\geq 1$ be the degree of the polynomial $g(x)$ and also, let A be the leading coefficient of g. We consider the following limit:

$$L:=\lim_{n\to\infty}\frac{g\left(c^{k_n}\right)}{c^{d\cdot k_n}}.$$

From basic calculus, it's clear that $L=A$ since $c^{k_n}\to\infty$ as $n\to\infty$ (note that $c>1$). On the other hand, we have that

$$L=\lim_{n\to\infty}c^{\ell_n-dk_n}$$

and so, the limit L exists and is *nonzero* if and only if there exists some integer b such that for all n sufficiently large, we have that

$$\ell_n-dk_n=b \tag{1}$$

(note that $c>1$ and so, powers of c won't accumulate near a nonzero real number). Hence $A=c^b$, but moreover, using also (1), we have that for each $x_n:=c^{k_n}$, where n is sufficiently large,

$$g(x_n)=Ax_n^d.$$

So, the polynomial $h(x):=g(x)-Ax^d$ vanishes at each point x_n (for n sufficiently large) thus showing that h must be identically equal to 0 (again note that the points x_n are distinct because $c>1$). So, always we have that

$$g(x)=c^b\cdot x^d\text{ for some }b\in\mathbb{Z}.$$

Problem 4. Let $f:[0,1]\longrightarrow\mathbb{R}$ be a function whose derivative is continuous, which also satisfies $\int_0^1 f(x)\mathrm{dx}=0$. Prove that for each $\alpha\in(0,1)$ we have

$$\left|\int_0^\alpha f(x)\mathrm{dx}\right|\leq\frac{1}{8}\cdot\max_{0\leq x\leq 1}|f'(x)|.$$

Solution. We define the function $g : [0,1] \longrightarrow \mathbb{R}$ given by

$$g(x) := \int_0^y f(y)\mathrm{dy}.$$

Then $g(0) = g(1) = 0$ and clearly, $g(x)$ is a function whose derivative (which is $f(x)$) is continuous. So, there exists a point - call it α - inside the interval $(0,1)$ with the property that

$$\left|\int_0^\alpha f(x)\mathrm{dx}\right| \text{ is the largest.}$$

Then $x = \alpha$ is a critical point for the function g and thus,

$$0 = g'(\alpha) = f(\alpha).$$

So, since the maximum is attained at $x = \alpha$, it suffices to prove that

$$\left|\int_0^\alpha f(x)\mathrm{dx}\right| \leq \frac{M}{8},$$

where $M := \max_{0\leq x\leq 1} |f'(x)|$.

We may assume that $\alpha \leq \frac{1}{2}$ since otherwise we may replace $f(x)$ by $f(1-x)$ which leaves our hypotheses unchanged, while M would still be unchanged and also,

$$\max_{0\leq y\leq 1}\left|\int_0^y f(x)\mathrm{dx}\right|$$

would be unchanged, but this time α would be replaced by $1-\alpha$. So, from now on, we assume $\alpha \leq \frac{1}{2}$.

Without loss of generality, we may assume that

$$\int_0^\alpha f(x)\mathrm{dx} > 0$$

since otherwise we could just replace $f(x)$ by $-f(x)$ and still prove the same conclusion.

Now, because $f(\alpha) = 0$ and $f'(x) \geq -M$, we conclude that

$$f(x) \leq M(\alpha - x) \text{ for each } 0 \leq x \leq \alpha.$$

So, since we also argued that we may assume that $\alpha \leq \frac{1}{2}$, then we have:

$$\left|\int_0^\alpha f(x)\mathrm{dx}\right| = \int_0^\alpha f(x)\mathrm{dx} \leq \int_0^\alpha M(\alpha - x)\mathrm{dx} = \frac{M\alpha^2}{2} \leq \frac{M}{8}.$$