PUTNAM PRACTICE SET 11

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Problem 1. Let $p > 3$ be a prime number. Prove that at least one of the numbers from the following list:

$$\frac{3}{p^2}, \frac{4}{p^2}, \frac{5}{p^2}, \dots, \frac{p-2}{p^2}$$

can be written as a sum $\frac{1}{x} + \frac{1}{y}$ for some positive integers x and y.

Solution. Let i be a divisor of $p+1$ other than 1, 2 or p; such a divisor exists since $p+1$ is a composite number larger than 4. Then let $\ell := \frac{p+1}{i}$; this is also a positive integer. Then

$$\frac{1}{p\ell} + \frac{1}{p^2\ell} = \frac{p+1}{p^2\ell} = \frac{i}{p^2},$$

as desired.

Problem 2. If $r > s > 0$ and $a > b > c > 0$, prove that

$$a^r b^s + b^r c^s + c^r a^s \geq a^s b^r + b^s c^r + c^s a^r.$$

Solution. We divide by c^{r+s} and let $x := a/c$ and $y := b/c$; then $x > y > 1$ (while $r > s > 0$). The inequality reduces to

$$x^r y^s + y^r + x^s \geq x^s y^r + y^s + x^r$$

and therefore, letting $t := r/s > 1$, while $u := x^s$ and $v := y^s$, we need to prove that

$$u^t v + v^t + u > uv^t + v + u^t.$$

This translates into proving the inequality

$$\frac{u^t - 1}{u - 1} > \frac{v^t - 1}{v - 1}$$

for all $u > v > 1$ and all $t > 1$. The above inequality is obvious using that the function $f(u) := u^t$ is concave up for $u > 1$ and $t > 1$; note that the two sides of the above inequality represent slopes of secant lines after connecting the point $(1, 1)$ with the points (u, u^t), resp. (v, v^t) on the graph of the function f.

Problem 3. Find all $f \in \mathbb{C}[x]$ with the property that for each $x \in \mathbb{C}$, we have $f(x)f(2x^2) = f(2x^3 + x)$.

Solution. Assume f is not identically equal to 0 or 1; then f cannot be a constant polynomial; so, it must have finitely many complex roots. We start with an easy claim.

Claim 1. If $\alpha \in \mathbb{C}$ is a root of $f(x)$, then also $2\alpha^3 + \alpha$ must be a root of $f(x)$.

Indeed, $f(\alpha) = 0$ and since $f(x)f(2x^2) = f(2x^3 + x)$, we get that $f(2\alpha^3 + \alpha) = 0$, as claimed.

The next claim is also simple.

Claim 2. Let $z_0 \in \mathbb{C}$ and let $z_1 = 2z_0^3 + z_0$.

(A) If $|z_0| > 1$, then $|z_1| > |z_0|$.
(B) If $|z_0| = 1$, then $|z_1| \geq 1$ with equality if and only if $z_0 = \pm i$.

For the proof of **Claim 2**, we note that if $|z_0| \geq 1$ then

$$|z_1| = |z_0| \cdot |2z_0^2 + 1| \geq |z_0| \cdot \left(2|z_0|^2 - 1\right) \geq |z_1|$$

with equality if and only if $|z_0| = 1$ and moreover, 1 and $2z_0^2$ have arguments which are opposite angles, i.e., the argument of z_0^2 must be π, which yields that the argument of z_0 must be either $\pi/2$ or $3\pi/2$. So, the equality is achieved only if $z_0 = \pm i$, as claimed.

The next claim is one of the two crucial observations.

Claim 3. There is no complex root of $f(x)$ of absolute value larger than 1.

Indeed, if there were such a root z_0, then letting $g(z) := 2z^3 + z$, we have that $z_1 := g(z_0)$ is also a root of $f(x)$ of absolute value larger than $|z_0|$ (according to **Claims** 1 and 2) and inductively, for each $n \in \mathbb{N}$ we have that $z_n := g^{\circ n}(z_0)$ must be a root of $f(x)$ and all these roots have increasing absolute value, which contradicts the fact that f has only finitely many roots. So, indeed, $f(x)$ has no root of absolute value strictly larger than one.

The other key piece of our argument is the following claim.

Claim 4. $f(0) \neq 0$.

Assuming that $f(0) = 0$, we let $1 \leq k \leq d := \deg(f)$ be the multiplicity of the root $z = 0$ for f; so, $f(x) = x^k \cdot h(x)$ where $h(0) \neq 0$. Then the equation $f(x)f(2x^2) = f(2x^3 + x)$ yields

$$x^{3k} \cdot 2^k h(x)h(2x^2) = x^k \cdot (2x^2 + 1)^k h(2x^3 + x)$$

and thus, $h(2x^3 + x) \cdot (2x^2 + 1)^k = x^{2k} \cdot 2^k h(x)h(2x^2)$, which contradicts the fact that $h(0) \neq 0$ (and that $k \geq 1$). So, indeed, $f(0) \neq 0$.

Now, **Claim** 4 yields that for the polynomial

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$$

we have that $a_0 = f(0) \neq 0$. On the other hand, since $f(0)^2 = f(0)$ (according to our hypothesis), then we must have that $a_0 = 1$. Furthermore, using again the identity $f(x)f(2x^2) = f(2x^3 + x)$ (but this time not for a specific value, such as we did above when we plugged in $x = 0$, however this time looking at the leading coefficient in the above identity of polynomials), we get that

$$a_d \cdot a_d 2^d = a_d 2^d \text{ and so, } a_d = 1.$$

In conclusion, the product of all the roots of $f(x)$ must have absolute value equal to 1. Now, **Claim** 3 yields that all roots have absolute value at most one and therefore, each root must have absolute value exactly one (since otherwise their product would not have absolute value 1). Now, for each root z_0 of $f(x)$, we have that $z_1 := 2z_0^3 + z_0$ is also a root of $f(x)$ (by **Claim** 1) and then using that $|z_0| = 1 = |z_1|$, **Claim** 2 yields that $z_0 = \pm i$. Finally, because $f \in \mathbb{R}[x]$, then the multiplicity of the root i in $f(x)$ is the same as the multiplicity of the root $-i$ and so, $d = 2k$ for some $k \in \mathbb{N}$ and we must have that $f(x) = (x^2 + 1)^k$. Finally, we check that this polynomial $f(x)$ (for any $k \in \mathbb{N}$) satisfies the desired polynomial identity:

$$\begin{aligned} &f(x)f(2x^2) \\ &= (x^2+1)^k \cdot \left((2x^2)^2+1\right)^k \\ &= (x^2+1)^k \cdot (4x^4+1)^k \\ &= (4x^6+4x^4+x^2+1)^k \\ &= \left((2x^3+x)^2+1\right)^k \\ &= f(2x^3+x), \end{aligned}$$

as desired.

Problem 4. Let $n \in \mathbb{N}$ and let $S_n = \{1, \ldots, n\}$. Assume the set $M \subseteq S_n \times S_n$ satisfies the following properties:

- if $(j, k) \in M$ then $1 \leq j < k \leq n$; and
- if $(j, k) \in M$ then for each $i \in S_n$, we have that $(k, i) \notin M$.

What is the largest possible cardinality of the set M?

Solution. We let

$$A := \{i \in S_n : \text{ there exists } j \in S_n \text{ such that } (i, j) \in M\}$$

and

$$B := \{j \in S_n : \text{ there exists } i \in S_n \text{ such that } (i, j) \in M\}.$$

By the hypothesis, we have that A and B are disjoint subsets of S_n. Also, for each $(i, j) \in M$ we have that $i \in A$ and $j \in B$. Therefore, $\#M \leq \#A \cdot \#B$ and clearly, the maximum possible is attained when $A \cup B = S_n$ **and** $\#A = \#B$ or $|\#A - \#B| = 1$. So, depending whether $n = 2k$, or $n = 2k + 1$, we have that the maximum number of elements in M is either k^2, or respectively $k^2 + k$. Also, we note that we can indeed construct sets M with such cardinality by considering

$$M = \{1, 2, \ldots, k\} \times \{k + 1, k + 2, \ldots, 2k\}$$

if $n = 2k$, while if $n = 2k + 1$, then we can simply take

$$M = \{1, 2, \ldots, k\} \times \{k + 1, k + 2, \ldots, 2k + 1\}.$$