PUTNAM PRACTICE SET 7

PROF. DRAGOS GHIOCA

Problem 1. Let $a, b, c, d \in \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $f(x) = 1 - a\cos(x) - b\sin(x) - c\cos(2x) - d\sin(2x)$. If $f(x) \geq 0$ for each $x \in \mathbb{R}$, then prove that $a^2 + b^2 \leq 2$ and also that $c^2 + d^2 \leq 1$.

Problem 2. Find all positive integers n for which there exist nonzero polynomials $f, g \in \mathbb{Z}[x_1, \dots, x_n]$ such that

$$(x_1 + x_2 + \cdots + x_n) \cdot f(x_1, \dots, x_n) = g(x_1^2, \dots, x_n^2).$$

Problem 3. Let $n \geq 3$ be a positive integer and let S_n be the set of all integers of the form $1 + kn$ for some $k \in \mathbb{N}$. We say that a number $m \in S_n$ is indecomposable if there exist no $x, y \in S_n$ such that $m = xy$. Prove that there exists some $s \in S_n$ which can be written in at least two distinct ways as a product of indecomposable numbers from S_n (note that two decompositions consisting of precisely the same indecomposable numbers, but appearing in a different order are considered to be the same decomposition).

Problem 4. Let n be an integer ≥ 2. We define two sequences $\{x_i\}_{1 \leq i \leq n}$ and $\{y_i\}_{1 \leq i \leq n}$ given by:

$$x_1 = n,\ y_1 = 1,\ x_{i+1} = \left[\frac{x_i + y_i}{2}\right] \text{ and } y_{i+1} = \left[\frac{n}{x_{i+1}}\right],$$

where $[z]$ is the integer part of z for each real number z. Prove that

$$\min_{i=1}^{n} x_i = [\sqrt{n}].$$