PUTNAM PRACTICE SET 6

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Problem 1. Find the largest possible integer which is the product of finitely many positive integers whose sum equals 2018.

Problem 2. Let $P \in \mathbb{R}[x]$ be a polynomial with the property that $P(x) > 0$ for each positive real number x. Then prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ with all coefficients nonnegative, such that $P = \frac{Q_1}{Q_2}$.

Problem 3. Prove that there exist infinitely many $n \in \mathbb{N}$ with the property that 7^n contains in its decimal expansion 2018 consecutive digits equal to 0.

Problem 4. Let $a \in (0, 1)$ be a real number. We consider the function $f : (0, 1] \longrightarrow (0, 1]$ given by:

$$f(x) = x + 1 - a \text{ if } 0 < x \leq a \text{ and } f(x) = x - a \text{ if } a < x \leq 1.$$

Prove that for any interval $I \subseteq (0, 1]$, there exists a positive integer n such that $f^{\circ n}(I) \cap I \neq \emptyset$.

Problem 5. Find (with proof) all possible function $f : \mathbb{N} \longrightarrow \mathbb{N}$ with the property that $f(n+1) > f(f(n))$ for each $n \in \mathbb{N}$.