PUTNAM PRACTICE SET 4

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Problem 1. Let $f(x) = x^2 - 2$. For each $n \in \mathbb{N}$, we let $f^{\circ n} := f \circ f \circ \cdots \circ f$ (n times). Prove that for each $n \in \mathbb{N}$ there exist 2^n real numbers x such that $f^{\circ n}(x) = x$.

Problem 2. Prove that there exists an infinite set S of points on the unit circle of radius 1 with the property that the distance between any two points from the set S is a rational number.

Problem 3. Consider the sequence $\{x_n\}_{n\geq 0}$ given by:

$$x_0 = 5 \text{ and } x_{n+1} = x_n + \frac{1}{x_n} \text{ for all } n \geq 0.$$

Prove that $45 < x_{1000} < 45.1$.

Problem 4. Let $n \in \mathbb{N}$ and let $a_0, a_1, \dots, a_{n+1} \in \mathbb{R}$ such that $a_0 = a_{n+1} = 0$ and $|a_{k-1} - 2a_k + a_{k+1}| \leq 1$ for each $k = 1, \dots, n$. Prove that for each $k = 0, \dots, n+1$, we have $|a_k| \leq \frac{k(n+1-k)}{2}$.

Problem 5. A sequence $\{x_n\}_{n\geq 0}$ is defined as follows:

$$x_0 = 2,\ x_1 = \frac{5}{2} \text{ and for each } n \geq 1, \text{ we have } x_{n+1} = x_n(x_{n-1}^2 - 2) - x_1.$$

Prove that for each $n \in \mathbb{N}$, we have that the integer part of x_n (denoted by $[x_n]$) equals $2^{\frac{2^n - (-1)^n}{3}}$.

Problem 6. Let $m \in \mathbb{N}$. We consider the m-by-$2m$ matrix

$$A = (a_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq 2m}}$$

with the property that each entry $a_{i,j}$ is either -1, 0, or 1. Prove that there exist integers $x_1, \dots, x_{2m}$ not all equal to 0 but also satisfying the inequality $|x_i| \leq m$ for each $i = 1, \dots, m$ such that

$$\sum_{j=1}^{2m} a_{i,j}x_j = 0 \text{ for each } i = 1, \dots, m.$$