PUTNAM PRACTICE SET 31

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Problem 1. Is there an infinite sequence of real numbers $\{a_n\}_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} a_n^m = m$$

for each $m \in \mathbb{N}$?

Problem 2. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be placed into k boxes with the sum of the integers in each box being the same across all boxes?

Problem 3. Find all differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for each $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$.

Problem 4. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with continuous partial derivatives, which satisfies the following equation:

$$f(x, y) = a \cdot \frac{\mathrm{df}}{\mathrm{dx}}(x, y) + b \cdot \frac{\mathrm{df}}{\mathrm{dy}}(x, y)$$

for each $(x, y) \in \mathbb{R}^2$. Prove that if there exists a constant M such that

$$|f(x, y)| \leq M \text{ for each } (x, y) \in \mathbb{R}^2,$$

then f must be identically equal to 0.