PUTNAM PRACTICE SET 3

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Problem 1. The set A has 6 elements. Determine with proof whether we can find distinct subsets $B_1, \dots, B_m$ of A (for some $m \in \mathbb{N}$) satisfying the following properties:

- each B_i has exactly 3 elements;
- for any two elements x and y of A, there exist precisely two indices $1 \le i < j \le m$ such that $x, y \in B_i$ and $x, y \in B_j$.

Problem 2. We consider the following real numbers:

$$x_1 \le x_2 \le \cdots \le x_n \text{ and } y_1 \le y_2 \le \cdots \le y_n.$$

Let $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ be any permutation. Prove that

$$\sum_{i=1}^{n} (x_i - y_i)^2 \le \sum_{i=1}^{n} \left(x_i - y_{\sigma(i)}\right)^2.$$

Problem 3. For any given positive real numbers a, b, c, d, we let

$$S_{a,b,c,d} := \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

Find all possible values that are taken by $S_{a,b,c,d}$ as we vary a, b, c, d in the set of all positive real numbers.

Problem 4. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of rela numbers satisfying the following properties:

(i) $0 \le a_n \le 1$ for each $n \ge 1$; and
(ii) $a_n - 2a_{n+1} + a_{n+2} \ge 0$ for each $n \ge 1$.

Prove that for each $n \ge 1$, we have that $0 \le (n+1)(a_n - a_{n+1}) \le 2$ for each $n \ge 1$.

Problem 5. Let M be the set of all positive integers which do not contain the digit 9 in their decimal expansion. Prove that $\sum_{x \in M} \frac{1}{x} < 80$.

Problem 6.

(A) Show that the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

is not a rational number.

(B) Let a and b be integers larger than 1. We construct two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ as follows:

$$a_1 = a \text{ and for all } n \geq 1, \text{ we have } a_{n+1} = na_n - 1$$

and

$$b_1 = b \text{ and for all } n \geq 1, \text{ we have } b_{n+1} = nb_n + 1.$$

Prove that there exist at most finitely many pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that $a_m = b_n$.