PUTNAM PRACTICE SET 28

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Problem 1. What is the maximum number of points in the cartesian plane whose both coordinates are rational numbers, which lie on the same circle whose center is not a point whose both coordinates are rational numbers?

Problem 2. Let $F_0(x) = \log(x)$ and for each $n \geq 1$ and $x > 0$, we let

$$F_n(x) = \int_0^x F_{n-1}(t)\mathrm{dt}.$$

Compute

$$\lim_{n\to\infty} \frac{n! \cdot F_n(1)}{\ln(n)}.$$

Problem 3. Let p be a prime number and let $f \in \mathbb{Z}[x]$. Assume that the integers $f(k)$ for $0 \leq k \leq p^2 - 1$ are all distinct modulo p^2. Then prove that for each $n \in \mathbb{N}$, the integers $f(k)$ for $0 \leq k \leq p^n - 1$ are distinct modulo p^n.

Problem 4. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose derivative is continuous with the property that for each rational number $\frac{a}{b}$, written in lowest terms (i.e., $a, b \in \mathbb{Z}$ with $b \in \mathbb{N}$ and $\gcd(a, b) = 1$), we have that also $f\left(\frac{a}{b}\right)$ is a rational number whose denominator, when we write $f(a/b)$ in lowest terms, is also equal to b.