PUTNAM PRACTICE SET 26

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Problem 1. Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence given by

$$a_1 = 1 \text{ and } a_{n+1} = 3a_n + \left[\sqrt{5}\cdot a_n\right] \text{ for } n \geq 1.$$

Compute a_{2021}.

Problem 2. Let $n \in \mathbb{N}$. Find the number of pairs of polynomials $(P(x), Q(x)) \in \mathbb{R}[x] \times \mathbb{R}[x]$ satisfying the following two conditions:

- $\deg(P) > \deg(Q)$; and
- $P^2(x) + Q^2(x) = x^{2n} + 1$.

Problem 3. Let $k \in \mathbb{N}$. Prove that there exist polynomials $P_0, P_1, \dots, P_{k-1}$ (which may depend on k) with the property that for each $n \in \mathbb{N}$, we have

$$\left[\frac{n}{k}\right]^k = P_0(n) + P_1(n)\cdot\left[\frac{n}{k}\right] + P_2(n)\cdot\left[\frac{n}{k}\right]^2 + \cdots + P_{k-1}(n)\cdot\left[\frac{n}{k}\right]^{k-1},$$

where (as always) $[x]$ is the integer part of the real number x.

Problem 4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with the property that

$$f(x,y) + f(y,z) + f(z,x) = 0,$$

for all real numbers x, y and z. Prove that there must exist another function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x,y) = g(x) - g(y),$$

for all real numbers x and y.