PUTNAM PRACTICE SET 25

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Problem 1. Let $n \in \mathbb{N}$ and let $a_1, \dots, a_n \in \mathbb{R}$. Show that there exists an integer m and some nonempty subset $S \subseteq \{1, \dots, n\}$ with the property that

$$\left| m + \sum_{i \in S} a_i \right| \le \frac{1}{n+1}.$$

Problem 2. For each continuous function $f : [0,1] \longrightarrow \mathbb{R}$, let

$$I(f) := \int_0^1 x^2 f(x) \mathrm{dx} - \int_0^1 x f(x)^2 \mathrm{dx}.$$

Find the maximum of $I(f)$ over all possible continuous functions f.

Problem 3. Let c be a real number greater than 1 and let $g \in \mathbb{R}[x]$ be a non-constant polynomial with the property that there exists an infinite sequence $\{k_n\} \subseteq \mathbb{N}$ with the property that for each $n \ge 1$, we have that there exists some $\ell_n \in \mathbb{N}$ with the property that

$$g\left(c^{k_n}\right) = c^{\ell_n}.$$

Find all such polynomials g.

Problem 4. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function whose derivative is continuous, which also satisfies $\int_0^1 f(x) \mathrm{dx} = 0$. Prove that for each $\alpha \in (0,1)$ we have

$$\left| \int_0^\alpha f(x) \mathrm{dx} \right| \le \frac{1}{8} \cdot \max_{0 \le x \le 1} |f'(x)|.$$