PUTNAM PRACTICE SET 23

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Problem 1. Show that each positive integer can be written as a sum of integers of the form $2^a \cdot 3^b$ with the property that no interger from the chosen sum divides a different integer from the sum.

Problem 2. Let $n \in \mathbb{N}$ and let $P \in \mathbb{C}[z]$ be a polynomial of degree $2n$, all of whose roots have absolute value equal to 1. Let

$$g(z) := \frac{P(z)}{z^n}.$$

Prove that each solution for $g'(z) = 0$ (where g' is the derivative of g) has absolute value equal to 1.

Problem 3. Let A be an N-by-N matrix with the property that each one of its entries is equal to 1 or -1 and also satisfying that $A \cdot A^t = N \cdot \mathrm{id}_N$ (where id_N is the N-by-N identity matrix). Assume there exists an a-by-b submatrix of A whose entries are all equal to 1. Prove that $ab \leq N$.

Problem 4. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} \mathrm{dx}$$